Problem Solving and Mathematical Modelling: Applicable Mathematics

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Abstract

Problem solving is often presented as a key skill possessed by mathematics students. An important question is, therefore, how students can best develop their problem solving skills during their mathematics education? From the student's perspective, problem solving is often viewed in terms of the ability to answer exam-style questions where the general form of the questions is familiar to the students. Employers and lecturers, on the other hand, regard problem solving as the act of modelling and then solving unstructured problems. We sought to address these issues by introducing problem solving with contextual problems, then progressing on to problems that require a qualitative rather than quantitative analysis, before finally developing the students' modelling skills. Keele University's Mathematics Department developed a new first year module which aimed to develop these skills and use innovative methods that allow students to express their creativity.

Background and Rationale

Keele University Mathematics Department has extensive experience dealing with the issues undergraduates face during the transition from the school-style mathematics presentation and the more contextual-style of mathematics seen in real-life problems. A simple example helps illustrate the problems many students face:

> `If a ball is dropped from 10 metres and bounces half the height it falls on each bounce, what is the total distance the ball travels until it is stationary?'

Although students have seen geometric series and their infinite sum at A-level they face two distinct challenges with questions like the above:

- They question is not familiar and does not fit into a pattern of similar problems;
- When asked to draw a diagram to illustrate this problem the students are unable to visualise the problem. A common error is for the students to ignore the ball bouncing up and then back down again.

Thus, although the techniques required when solving the problem are well within a student's repertoire of mathematical tools they are unable to convert the contextual problem into a mathematical representation. We would argue that the students are unable to implement part of the modelling cycle; that is, from setting up a model to developing a mathematical problem. This realisation meant the starting point for our new module should be `unstructured problem solving'. The students required a solid grounding in developing conceptualisation skills before we could think about assumptions, parameters, variables and the modelling cycle. Indeed it was hoped that we could build the following skills in the students;

- Visualisation of `word' problems;
- Selection and application of previous knowledge to `word' problems;
- Presentation of mathematical arguments to peers and professionals;
- Present mathematics in creative ways to peers and professionals;

- Qualitatively (rather than quantitative) analyse problems;
- Appreciate the importance of the modelling cycle;
- Apply the modelling cycle to population problems.

From a pedagogical viewpoint we wanted the students to actively engage with the development of their learning. The development of the module as based around a student-, rather than educator-, centric perspective. It is well-known that a student who actively engages in the learning process uses higher-level cognitive domains and has a better learning experience (Ramsden, 2007, p.46).

Implementation

Keele University's Mathematics department introduced a new first year module, Applicable Mathematics. This module was a second semester option for single honours mathematics students. During the initial run 36 students registered for the module.

The module assessment was divided as follows:

- Three group projects contributed 70% to the final mark.
- Three individual assessments which contributed 20% to the final mark.
- A class participation mark of 10%.

As mentioned previously a key aim for the module was student creativity. It was decided that the students would make use of the following during the assessment process:

- Wikis. A short video was created by the module leader to demonstrate how to use the VLE's wikis. The students at various points during the module were required to create Wikis and present them to their peers;
- Posters. The main outputs of two group projects were posters. These were professionally printed. The posters were presented to their peers and members of the department and are now displayed in the department as examples of student work.
- Video cameras. Students had to produce videos demonstrating their results. A video camera was purchases for this. A selection of links to student videos can be found in Appendix 1.
- Reports. Students had to produce reports during the module. These included a report on the mathematics of juggling, a qualitative analyse of a problem and a final modelling report.

The module was divided into three parts each focusing on a different aspect of the students' development. The module was timetables as follows during the week:

- Monday 1 hour class
- Thursday 1 hour class
- Friday 2 hour class.
  This was to allow problems to be set at the start of the week and for the students to prepare material for the later session in the week.

Below we give a brief summary of the module, a detailed description can be found in Appendix 2.

Part 1: Unstructured Problem Solving.

The students engaged in group problem solving activities. The problems were word-based problems allowing the students to develop several key skills: conceptualisation, analysing and application. The problems required no mathematical tools beyond GCSE or A-level, but the students had to use higher-level cognitive domains to solve the problems.

The main assessment of this part of the module was via a group project. The students were given an unstructured word problem. They were required to solve and then present their solution using posters and also a video. The posters and videos were assessed by their peers.

Part 2: Making assumptions.

The second part of the module examined making assumptions to simplify problems. The students, once again, worked in groups and were given a variety of problems that they had to simplify. Several of the problems had no definitive solution and were include so that students started to engage with the idea of “form”. Examples of such problems were “How many grains of sand are there on Earth?”; “What percentage of the world’s water is contained in a cow?”

The students had to work in groups to determine what they considered to be a qualitative answer to the given problems and produce written reports.

Part 3: Mathematical Modelling.

The students were introduced to discrete modelling of populations. Students were allocated to groups and each group was assigned an animal. The groups were required to research and develop a mathematical model for the growth of their allocated animal.

In real life animals interact with each other, typical either competing for resource, for food or indeed symbiotically. We introduce interactions between our chosen animals; these interactions corresponded to interactions between the different student groups. The students had to generalise their models to include the different interactions between their animal and animals it interacted with.

The students were required to generate a class poster that illustrated the outcomes from each group and the overall outcomes from the project. The class also produced a video to complement the poster that illustrated the outcomes of the project.

Over the 12 weeks the module ran we adopted the schedule shown in Table 1 (again see Appendix 2 for further details).

This academic year one member of staff ran the module. In future years scaling the module is not viewed as a problem; it is anticipated that the first years who took the module during this academic year will act as mentors in subsequent years.

This was a new module and Keele University required university module evaluation procedures to be adopted.

Evaluation

Two instruments were provided for evaluation purposes, these instruments are included in Appendix 3. It was intended that we would sample the student responses before and after they had taken the module. Unfortunately since the module finished on the final day of the semester the students did not return the second instrument on `Awareness of mathematical modelling’. For this reason the evaluation is based on student responses to the first instrument ` Your attitude to mathematics applications and mathematical modelling’.

At the start of the module the student responses were:

Table 1: Summary schedule for the Applicable Mathematics module.

Week	Lesson	Activity
1	1	Organise the students into their groups. Introduction to the module. Give out first set of group problems.
	2	Groups presented their solutions to their allocated problem.
	3 and 4	Group and mini relay round from UKMT STMC.
2	1	Review of problems that cause difficulty from Week 1, Lessons 3 and 4.
	2	Further practice of the mini-relay question.
	3 and 4	Group and cross number round from UKMT STMC.
3	1	Group Project 1 Introduction and start.
	2	Group Project 1
	3 and 4	Group Project 1
4	1	Group Project 1
	2	Group Project 1
	3 and 4	Group Project 1 (Deadline for posters so they could be professionally printed.)
5	1	Group Project 1
	2	Group Project 1 (Deadline for videos.)
	3 and 4	Group Project 1 Final Presentation.
6	1	Introduction to cipher methods. Series of challenges set.
	2	Discrete logs and Diffie-Hellman
	3 and 4	Group challenge.
7	1	Introductory problems on assumptions, variables and parameters.
	2	Group problem solving of a practical problem and presentation.
	3 and 4	Drug problem.
8	1	Group Project 2.
	2	Group Project 2.
	3 and 4	Group Project 2.
9	1	Introduction to modelling of populations.
	2	Predator-Prey, competition and symbiotic systems.
	3 and 4	Preliminaries for Group Project 3.
10	1	Group Project 3.
	2	Group Project 3.
	3 and 4	Group Project 3.
11	1	Group Project 3.
	2	Group Project 3.
	3 and 4	Group Project 3.
12	1	Group Project 3.
	2	Group Project 3.
	3 and 4	Group Project 3 Final Deadline.

Question	--	-	N	+	++	Don't Know
1	1	2	18	6		2
2	1	3	22	3		
3		4	19	6		
4		7	13	9		
5	10	11	6	1		1
6	10	11	6	1		1
7		8	16	4		1
8	2	4	21	1		1
9	3	11	12	3		
10	9	13	7			
11	10	12	6	1		

It is worth highlighting the students' negative attitudes to questions 5, 6, 10 and 11. Question 5 asked the students for their attitude to:

` Working in a team where you have to listen to others explain some mathematics'.

Question 6 asked for the students for their attitude to:

` Working in a team where you have to explain some mathematics to others'.

Question 10 asked the students their attitude to:

` Researching a situation which demands understanding some mathematics'.

Finally, Question 11 asked the students about their attitude towards:

` Writing a report of how you used mathematics for a non-specialist audience'.

It should not surprise us that the students have negative attitudes to these four key areas:

- Team work is not something they have had to engage in before.
- They have never had to explain mathematics to others. One might argue that writing exam answers is explaining mathematics to others, we would counter this by saying exam answers are often seen by students as something the educator has to decode rather than a professional presentation of material.
- Students would not have had to research material before.
- Students have not had to write mathematical reports.

By the end of the module the student responses to the same instrument were:

Question	--	-	N	+	++	Don't Know
1				28		1
2			2	25	2	
3			3	23	3	
4	1		6	20	2	
5				15	14	
6				12	17	
7			3	21	5	
8			8	19	2	
9			4	24	1	
10			1	13	15	
11				14	15	

In both cases the sample size was 29 students. This is below the number of registered students due to absence due to illness.

Students have responded favourable to questions 5, 6, 10 and 11. Moreover, there has been positive shift in the students' attitudes.

It is important to note that the students responses to this instrument were rushed (since it was their final session) and we feel the results may have been slightly positively biased due to them having just completed their final assessment.

In future to ensure a fair reflection and also completion of both instruments we would make completion part of the assessment process: the students would be reflecting on their learning over the module. We would also ask the students to complete the instruments a week after the end of the module to allow adequate reflection on their learning.

Discussion, Learning and Impact

The following are key outcomes from the project:

- problems that may be viewed by mathematicians as trivial are considered challenging by A-level students;
- the challenge for students lies not in the mathematics but in visualisation;
- students require plenty of practice of word-based problems before they can be expected to engage with the full modelling cycle;
- an increase in students' attitutude to working in teams, explaining mathematics and researching. These are skills that most first-year mathematicians lack;
- students present mathematics better when they are being assessed by their peers;
- when given the opportunity students can be incredibly creative;
- when students are placed in competition with each other, rather than against a mark scheme, the effort they put into their work increases dramatically.

The evaluation would suggest that students are more positive about their problem solving and modelling experiences after the module than before. In particular we would seek to emphasis the improved team work, research and report writing attitudes. A key tenant of the module was to ensure that all students could contribute and this appears to have been successful. In several cases students have been able to demonstrate skills that a typical examination-based module would not allow them to do. These skills included:

- organisation and team leadership skills;
- presentation skills;
- computer based skills;
- video creation and editing.

There are a few minor changes that we will be making to the module, just to tighten up the individual reporting in the group activities. We will also increase the work done using Wikis as this was viewed as a positive experience.

The key conclusions from this module are:

- We need to ensure that the students have sufficient time to develop the skills we wish them to;
- The idea of using video, although initially viewed as a potential barrier to success, worked very well. Students engaged well with this element. If a video camera is not available then the students are able to use their mobile phones and there is plenty of freely available video editing software. Although we purchased video editing software it turned out to be unnecessary.
- In future years we will have a pool of previous students to use as mentors. This will allow the module to scale to a larger numbers of students.
- The students enjoyed the competition element of some of their group activities.

Further Development and Sustainability

This project developed a complete 15 credit first-year mathematics module. The materials for this module will continued to be used for future iterations of this module and it will continued to be offered to first-year single honours undergraduates as an options. Discussion is underway to develop further modules with similar objectives so students can experience an "Applicable Mathematics" strand in our degree programme. In the first year it is envisaged that the current Applicable

Mathematics module will be split into two modules. The first module would be taught in Semester 1 and would be a core module for single honours mathematics students. This module would expand take the first five weeks on problem solving and expand it to a full module. The second module would be taught in Semester 2 and would be an option for single honour mathematics students. This module would expand on the modelling part of the current Applicable Mathematics module.

The materials from this project will be made available as part of the HE STEM program and can be freely used at other institutions.

Output

This project has developed a module on problem solving and mathematical modelling. The key outputs from the student perspective are the posters and videos. Examples of the videos are included in Appendix 1. Example posters are available in Appendix 4.

Appendix 2 contains a detailed description of where many of the problems used during the module can be found. In Appendix 5 we give examples of the types of problems used an introductory word problems and in the various group projects.

References

Ramsden, P. (2007). Learning to Teach in Higher Education, second edition, RoutledgeFalmer.

Appendix 1

As part of the first group project students were required to produce a video that complemented their project. Examples of these videos are available to view on YouTube using the following links:

- http://youtu.be/4b5tIjCr9QQ
- http://youtu.be/csfrLiFE0Mk
- http://youtu.be/gEy2OUf65WA
- http://youtu.be/llS5y8wcj2U
- http://youtu.be/d9IFOIQNAWM

The final group project again required the entire class to contribute to a video explaining the outputs of their modelling of an ecological food web. The video is available to view on YouTube using the following link:

- http://youtu.be/fN1YaKJA5ZI

Appendix 2

Section 1: Problem Solving

The aims of the first part of the module were to develop the following skills:

- Students' ability to work in teams to reach a joint outcome.
- Convert word-based mathematical problems into mathematics.
- Manage a project from inception to completion with different outcomes (poster and video).
- Write reports on meetings.

The following details the week-by-week student activities. This schedule is only suggested and should be adapted as required by the students.

Week 1

- Lesson 1:
    - Students were organised into group. The groups were allocated by the module leader. Allocation of groups was done by selecting the *n*th individual in the alphabetically list of students.
    - Students were given a quick introduction to the module.
    - Students were given a set of word-based problems to solve in groups.
    - At the end of the session each group was randomly allocated one problem. The students were then required to write up their problem for presentation to the rest of the class. It was decided that the students would write their presentations as a Wiki allowing the whole class free access to the material.
    - A YouTube video was provided on the VLE to illustrate the functionality of the Wiki. Students had no problems using the Wiki to create their content.
    - The deadline for the students to write-up their material was the following lesson.
- Lesson 2:
    - Students presented their solutions to their given problems;
    - As a class we discussed different aspects of the problems and alternative solutions.
    - If this session had gone quickly further problems had been prepared.
- Lessons 3 and 4:
    - This was a two hour session. To continue to encourage the key aims of this section a group competition was run over this two hour period.
    - During the first hour the students worked in groups on the Group Round questions from the United Kingdom Mathematics Trust's (UKMT) Senior Team Mathematics Challenge (STMC). Samples of these materials are available from the UKMT web page: http://www.mathcomp.leeds.ac.uk/team-challenges/senior-team-challenge/
    - The rules were exactly as for the STMC competition itself.
    - The students were given a short break during which time the Group Round answer were marked.
    - During the second hour the students competed in a mini-relay as describe on the UKMT web page: http://www.mathcomp.leeds.ac.uk/team-challenges/senior-team-challenge/. The following change was made:
        - Rather than working in pairs, the students worked as a group again. This was simply due to the logistical constraints.
    - The mini-relay was marked and the teams given their scores.
    - After the session a list of questions that proved difficult was complied. These questions were reviewed at the start of the following week.

Week 2

- Lesson 1:
  - Several key problems had been identified as causing problems for some groups. Those groups that had developed solutions presented them to the class.
  - Questions that had caused universal problems were discussed.
  - A set of back-up problems were available in case the two points above were not necessary.
- Lesson 2:
  - A set of UKMT STMC mini-relay problems were used.
  - In this case groups were given a set of questions (for example, set A). The students attempted to answer all the questions. They submitted their answers.
  - If their answers were correct then they were given the set of questions. (For example, set B).
  - If, however, students failed to answer a question correctly, they were given a small hint and then another attempt. This was repeated until they had successfully answered the problem set.
  - Unlike the original problems, no time limit was set. We were developing persistence at solving problems.

- Lessons 3 and 4:
  - This was a two hour session. To continue to encourage the key aims of this section a group competition was run over this two hour period.
  - During the first hour the students worked in groups on the Group Round questions from the United Kingdom Mathematics Trust's (UKMT) Senior Team Mathematics Challenge (STMC). Samples of these materials are available from the UKMT web page: http://www.mathcomp.leeds.ac.uk/team-challenges/senior-team-challenge/
  - The rules were exactly as for the STMC competition itself.
  - The students were given a short break during which time the Group Round answer were marked.
  - During the second hour the students competed in the cross-number as describe on the UKMT web page: http://www.mathcomp.leeds.ac.uk/team-challenges/senior-team-challenge/. The following change was made:
    - Rather than working in pairs, the students worked as a group again. This was simply due to the logistical constraints.
  - The cross number was marked and the teams given their scores.
  - After the session a list of problem questions was complied.

Weeks 3 to 5: Group Project 1.

During this period this period students worked in their groups on a single problem. Each group and individual was required to produce evidence of their work. This evidence consisted of:

- A group poster describing how they solved their problem.
- A group video describing their problem and solution.
- A series of group logs giving minutes of their minutes along with individual actions.
- Each individual in a group was required to produce an individual log detailing how they had addressed their actions from group meetings.
- *A key point is the students were told to produce material that they would be proud to present to family, friends and other students: the students were competing against each other rather than against a mark scheme.*

The posters and videos were assessed during a presentation on Friday of Week 9. This presentation consisted of:

- One room where the videos had placed on a repeating video play list.
- A separate room for the posters.
- Members of staff and students evaluated the posters and the videos, in particular:
  - Each group peer assessed each group's posters.

    - Each group peer assessed each group's video.
    - Each group was required to give detailed feedback on each group's poster and video together with a rating for each group's poster and video.
- Staff also provided feedback on the poster and videos, but the key element was the student-centred assessment.

Examples of the problems used during this stage are given in Appendix 2.

Section 2: Practical uses of mathematics and introduction to variables, parameters and assumptions.

The aims of the second part of the module were to develop the following skills:

- Develop students' appreciation for the importance of mathematics through practical examples.
- Making assumptions to simplify problems.
- Determine variables and parameters.
- Select appropriate techniques to solve problems.
- Draw appropriate conclusions.

The following details the week-by-week student activities. This schedule is only suggested and should be adapted as required by the students.

Week 6

- Lesson 1:
    - Introduction to ciphers and methods for sending information.
    - Groups were set a number of different code-based challenges.
    - Note: in subsequent years this will be changed to an introduction to networks and their applications. This is so the material will not overlap with a new module on ciphers and cryptography.

- Lesson 2:
    - Students were given an introduction to the discrete logarithm and the Diffie-Hellman method for generating shared keys.
    - A series of challenges were set for each group based on the Diffie-Hellman method.
    - Note: This will be changed in future to problems involving networks, in particular, mazes.
- Lessons 3 and 4:
    - This was a two hour session. The students were set two challenges to solve in groups;
    - The first challenge required them to send a message using only sight from the back of the room to the front.
    - The message needed to be sent quickly and accurately.
    - It was also required that the messages could not be understood by the other groups.

Week 7

The aim of this week's activities was to develop the students' ability to make assumptions, identify variables and parameter and develop simple models.

- Lesson 1:
    - The students were given the following problem:

      `If I could fold this piece of paper an infinite number of times then how many times would I need to fold it to reach the moon?'

    - The students needed to make a number of assumptions and qualitative judgements to answer the problem.
    - The different solutions and approaches to the problem were discussed.
- Lesson 2:
    - Each group was asked to investigate the following problem:

A wedding cake is to be baked in a square cake tin and will have a volume (before icing) of 4000cm^3. Determine the dimensions of the cake which will give the minimum surface area before icing (that is, the top and four sides).

Find the dimensions if the cake is backed in a circular tin.

There is a rule of thumb in cookery that one third of the marzipan should be used for the top of the cake and the remaining two thirds for the sides.

Investigate the validity of this rule of thumb?

- Each group made a presentation on the following:
  - The assumptions they made.
  - The variables and parameters.
  - Their mathematical model for the problem.
  - The initial conclusions from the model.
  - Evaluation of their approach.

- Lessons 3 and 4:
  - This was a two hour session. Each group was given the following problem:
    - Patients with asthma have constriction of the airways in the lungs and consequent difficulty in breathing out.
    - This ailment can be alleviated by introducing the drug theophylline into the bloodstream.
    - This is done by injecting another drug, aminophylline, which the body quickly converts to theophylline.
    - Once present in the blood, however, the drug is steadily excreted from the body via the kidneys. (Some people say the system `leaks' and unless there is replenishment the quantity of drug in the blood will fall.)
    - It is known from experiments that theophylline has hardly any therapeutic effect if its concentration in the bloodstream is below 5mg/l and that concentrations above 20mg/l are likely to be toxic.
    - Problem: To administer the drug in such a way that the concentration remains within the therapeutic range between 5mg/l and 20 mg/l.
  - Each group had to produce a presentation on the following:
    - The assumptions they made.
    - The variables and parameters.
    - Their initial mathematical model for the problem.
    - The initial conclusions from the model.
    - Evaluation of their approach and how their model could be improved.

Week 8: Group Project 2.

During this period students worked in their groups on a single problem. Each group and individual was required to produce evidence of their work. This evidence consisted of:

- A report submitted by the whole group on their approach to the problem;
- A series of group logs giving minutes of their minutes along with individual actions;
- Each individual in a group was required to produce an individual log detailing how they had addressed their actions from group meetings.

The problems were selected so there was no solution in the traditional sense: the students had to make assumptions to generate a qualitative understand of the problems. Such problems are:

- Sergey Bubka currently holds the pole vault world record at 6.14metres. This record was set on 31 July 1994 in Italy. Only seventeen men have ever exceeded 6 metres and experts generally believe that the best that could ever be achieved is 6.4 metres.
  Develop a simple model to explain potential heights that an athlete could reach.
- What percentage of the world's water is contained in a cow?

Section 3: Introduction to modelling of populations.

The aims of the third part of the module were to develop the following skills:

- Introduce students to modelling of ecological situations.
- Develop students' ability to use computers to visualise their models.
- Involve the entire class in a business-type problem-solving environment where different groups are required to work together to produce a single outcome.

The following details the week-by-week student activities. This schedule is only suggested and should be adapted as required by the students.

Week 9

- Lesson 1:
  - The students were given an introduction to modelling of population growth using difference equations.
  - Each group was given a different type of difference equations.
  - The groups had to program each model into Excel and report on their findings.

- Lesson 2:
  - An introduction to predator-prey, competition and symbiotic modelling.

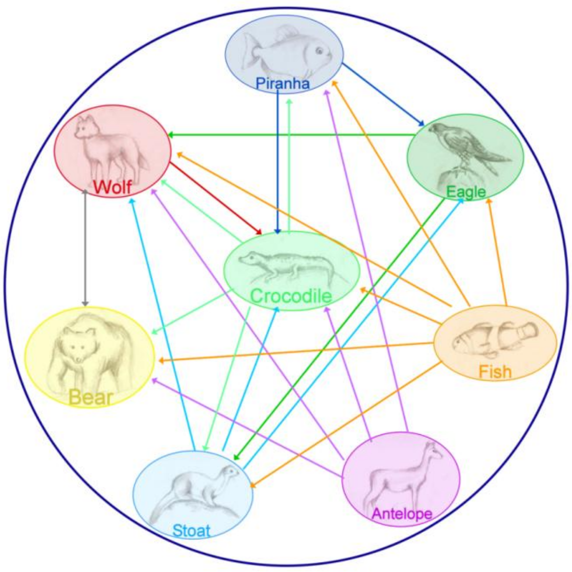
- Lessons 3 and 4:
  - Each group was allocated an animal from a selection of eight different animals.
  - The groups had to research their animal and develop a model to predict its growth.

Weeks 10 to 12: Group Project 3.

During this period students worked in their groups and class to develop a mathematical model for an ecological food web. Each group and individual was required to produce evidence of their work. This evidence consisted of:

- A class poster describing how they solved their problem.
- A class video describing their problem and solution.
- Each individual was required to produce a report on the mathematical model the class had developed. This report had to detail the problem, assumptions, variables and parameters, the mathematical models and the outcomes of the project.

The food web the students used is as follows:



The following should be noted:

- As the model grew more groups had to work with each other to develop the final model.
- Eventually the entire class was working as one group to generate the final model.

Appendix 3

Dear student,

This instrument measures the degree to which students feel 'positively' or 'negatively' about engaging in future experiences of mathematics and mathematical applications. It will be used to collect evidence of how the course influences students' dispositions.

Part A – About yourself and your course/university

Please complete the following questions about your institution and yourself, by filling in the boxes.

1	Name	
2	University name	Keele University
3	Course	Mathematics
4	Date	

Part B – Your attitude to mathematics applications and mathematical modelling

Please indicate how positive or negative you would feel about engaging in the activity/context mentioned by circling the appropriate symbol (eg -- for 'very negative', - for 'negative' etc.) or ticking the 'don't know' box (e.g. if the description doesn't make sense to you).

		Negative		N*	Positive		Don't know
1	Solving practical, real-life problems with mathematics.	--	-	N	+	++	
2	Reading and interpreting mathematics in a practical task	--	-	N	+	++	
3	Learning some new mathematics for use in your work	--	-	N	+	++	
4	Using mathematics to solve puzzles for leisure	--	-	N	+	++	
5	Working in a team where you have to listen to others explain some mathematics	--	-	N	+	++	
6	Working in a team where you have to explain some mathematics to others	--	-	N	+	++	
7	Devising a mathematical representation for a situation in order to solve a problem	--	-	N	+	++	
8	Drawing a diagram to model a situation when solving a problem	--	-	N	+	++	
9	Defining variables and parameters that might be important in a problem	--	-	N	+	++	
10	Researching a situation which demands understanding some mathematics	--	-	N	+	++	
11	Writing a report of how you used mathematics for a non-specialist audience	--	-	N	+	++	

**N stands for neutral; having neither negative nor positive feelings about the statement.*

Awareness of mathematical modelling

Part A – About yourself and your course/university

Please complete the following questions about your institution and yourself:

1	Name	
2	University name	Keele University
3	Course	Mathematics
4	Date	

Part B – Optional question for those students who have been involved with 'real problem solving using mathematics or mathematical modelling' in the past; if you feel this does not apply then proceed to the next Part, Part C

Briefly **describe** a substantial, real problem you have tackled using mathematics to model a situation and produce 'real' solutions. Avoiding too many details, emphasise:

- how you modelled the problem so that it became mathematical, (e.g. say what diagrams/variables/assumptions were important, how the problem was represented mathematically);

- what were important limitations of the model and so of the validity of solutions;

- how you interpreted, evaluated, compared or tested your mathematical solutions in practice (e.g. if the results were not good enough, why not, and what improvements to the model were considered, etc.)

Part C1 – if you have completed part B you may not have time to do this as well, but continue with this if you have time

A lifeguard is positioned at a point near the edge of a large pond/small lake, and wants to get to a swimmer near the edge of the pond but on its far side as fast as possible. A mathematical model is developed to give advice about whether the life guard should run round or swim across the pond, depending on the lifeguards' running and swimming speeds. The model developed makes some strict assumptions, and in fact predicts that running is better if the running speed is greater than about 1.5 times the swimming speed. (NB In this question you are not expected to actually calculate the answer, just describe what modelling might be involved.)

- Describe features of such a model (e.g. what diagrams/assumptions might need to be made, and what variables need to be introduced in formulating a simple, but sensible model?)

- Discuss limitations of any such model: what might be important limitations of the model and so of the validity of any solutions and predictions arising;

- How might the model's solutions/predictions be interpreted/evaluated/tested, (and how improved if found necessary)?

Part C2 – if you have completed part B you may not have time to do this as well, but continue with this if you have time

A couple plan to have at least one boy and one girl child, but want to know how large a family they may be likely to have in order to achieve this. A model of this situation involves representing the outcomes of each birth as equally likely, like the toss of a fair coin (Boys = tails, Girls = Heads, etc.) This model predicts that there is a half chance that they will have one of each after two births, a quarter chance after 3 births, an eighth chance after 4 births etc. (You are not expected here to make these calculations, but just to describe the model etc.)

- Describe important features of such a model (e.g. what diagrams/assumptions might be needed/made, and what variables need to be introduced in formulating such a simple, sensible model?)

- Discuss limitations of such a model: what might be important limitations of the model and so of the validity of solutions/predictions made;

  - How might the model's solutions/predictions be interpreted/evaluated/tested, (and how improved if found necessary)?

The criteria against which 'understandings' maybe could be mapped (at levels such as NONE, PARTIAL, GOOD, SOPHISTICATED):

1. Appreciates the nature of a 'mathematical model' as a descriptor/approximator/predictor/analogy of a real situation in which maths might help understanding

Eg simplifications/assumptions listed that may be only approximate: C1: diving in, slowing down at bends, the lake may not be circular, the swimmer may not be in the exact 'opposite position' assumed etc

2. Appreciates that models are always 'limited' by factors, such as the limited validity of their assumptions in practice

C1: thus diving in cuts a constant amount off the swimming time and this may be significant factor if the times are 'small enough'.

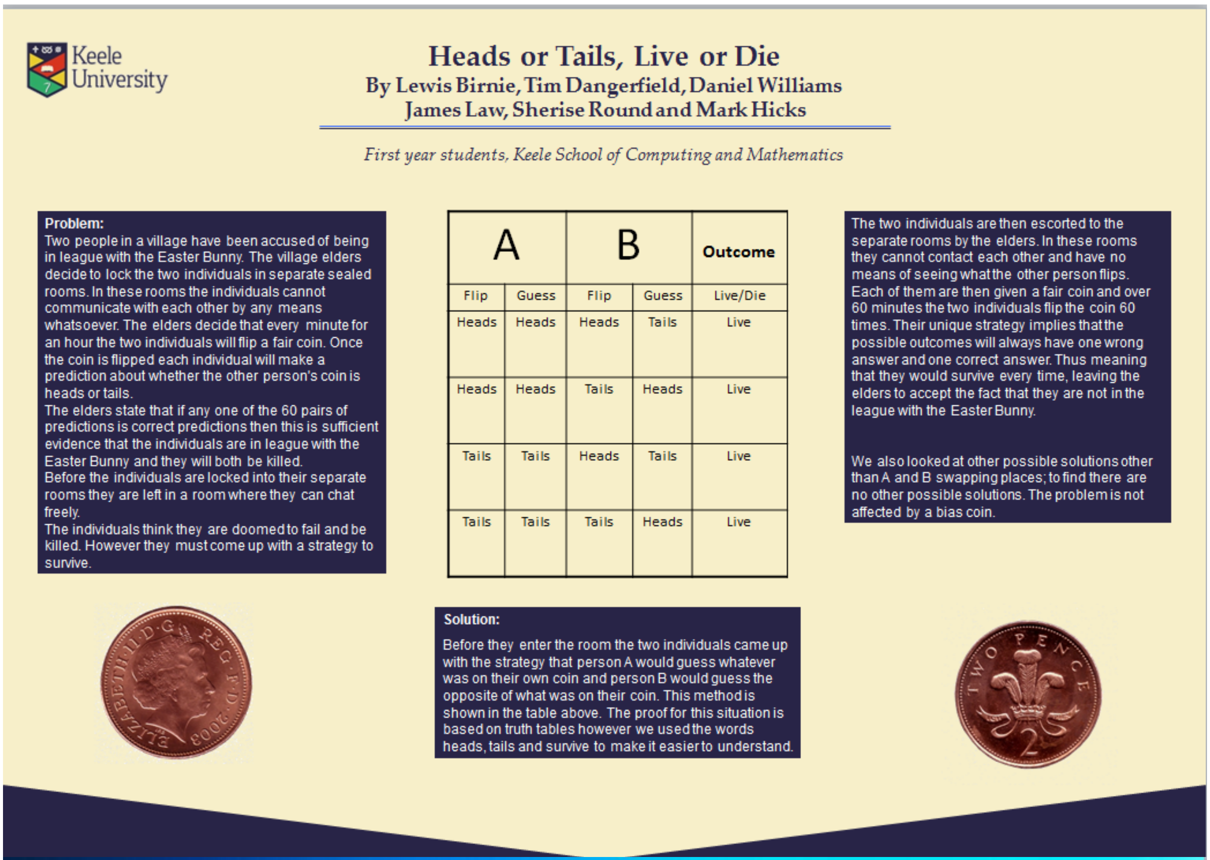
3. Appreciates that any model may need to be improved/ refined if they are found wanting/limited

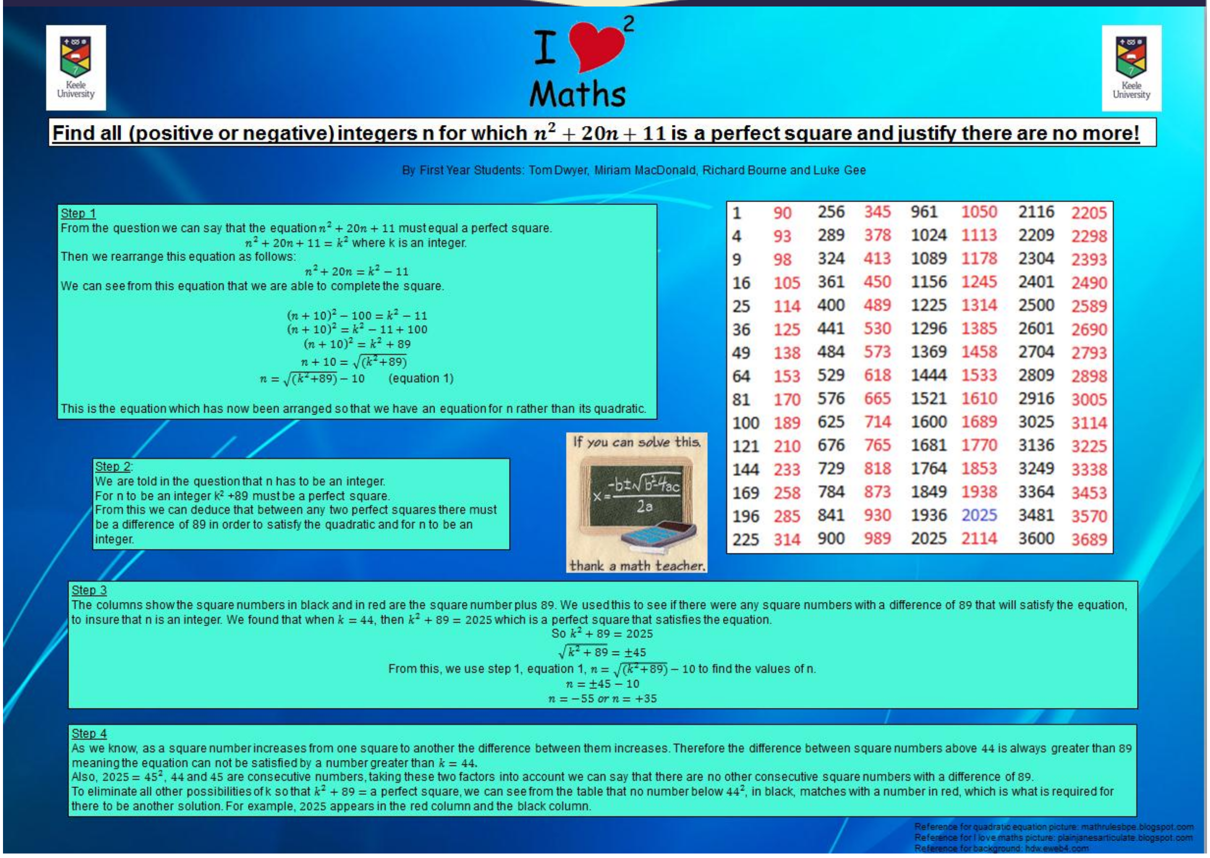
C1: By parameterising the position of the drowning swimmer, one may make different predictions depending on where the swimmer is placed in the pond.

4. Appreciates the 'modelling' process as one of mapping back and forth between a mathematical model and another context/situation: e.g. the 'real' criterion/problem needs to be crafted into a mathematical criterion

Appendix 4

The follow are examples of the posters students produced during the Applicable Mathematics module.



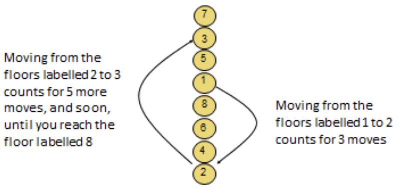


Problem 5: Lift Problem

Matthew Annetts, Emily Kruger, Michael Phillips, Graham Edwards
First Year Students, School of Computing and Mathematics, Keele University

The Problem

A lift in a building can move up and down between floors one floor at a time, each move, either up or down, counts as one move. The floors can be labelled in any way, however, the elevator still has to move between floors in numerical order. Considering a building of 8 floors labelled in the following way, the lift moves like shown;



Using this method, find the maximum number of moves possible for 8 floors, and the labelling of the floors which gives this maximum.

The Solution

The Maximum Number of Moves

To begin with we looked at buildings with fewer floors to try and see if a pattern emerged. We came up with the sequence (Un) 1,3,7,11,17,23 for $n \in N$, n>1. Since the differences were; 2,4,4,6,6, we decided to only consider the even terms, m=2n, since we only needed to prove the case for n=8. We then decided that writing it is an iterative formula, and from that predicting a formula to give U_m, which we would then prove using mathematical induction.

It is given that $U_m = U_{m-2} + 2m - 2$, $U_2 = 1$, and that m=2n $\forall n \in N$

Let P(n) be the proposition such that;

P(n): $U_m = \frac{1}{2}m^2 - 1$

In the case n=1, m=2;

P(1): $U_2 = \frac{1}{2}2^2 - 1 = 2 - 1 = 1$

Now assume the case for n=k, m=2k, that is;

P(k): $U_2 = \frac{1}{2}(2k)^2 - 1 = 2k^2 - 1$

Hence prove the case for n=k+1, m=2k+2,

It is given that:

$U_{2k+2} = U_{2k} + 2(2k+2) - 2 = U_{2k} + 4k + 2$

But, $U_{2k} = 2k^2 - 1$

So, $U_{2k+2} = 2k^2 - 1 + 4k + 2 = 2k^2 + 4k + 1 = \frac{1}{2}(2k+2)^2 - 1$

Hence, P(1) is true and if P(k) is true then P(k+1) is true. Hence by the theory of mathematical induction P(n) is true $\forall n \in N$

Therefore $U_8 = \frac{1}{2}(8)^2 - 1 = 32 - 1 = 31$

So, 31 is the maximum number of moves the lift can make for 8 floors.

The Labelling of the Floors

From experimenting with different floor combinations we spotted a couple of patterns which needed to be met to give the optimum solution of 31, these were:

- The labels '1' and '8' needed to be next to each other (in the middle)
- Even and odd numbers needed to be separated so that there are no even numbers next to an odd one (with the exception of '1' and '8')

The other labels can be arranged in any order as long as these conditions are met giving the optimum solution. This gives a total of 72 different combinations out of a possible 40320.

Pick A Door...

Kate Broad, Dan Horton, Matt McQuay, James Watson, Mike Williams
1st Year Students, Department of Computing and Mathematics, Keele University

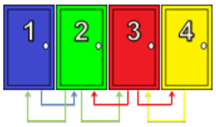
You are a contestant on a game show...

...and there are 4 doors to choose from behind which is one prize. The task is to pick a door: If the prize is behind that door, you win! If the prize is behind a different door then the host's assistant moves the prize as follows:

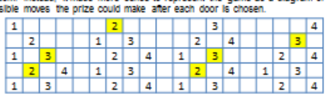
- If the prize is behind door 1; it is moved behind door 2.
- If the prize is behind door 2; it is moved either behind door 1 or behind door 3,
- If the prize is behind door 3; it is moved either behind door 2 or behind door 4,
- If the prize is behind door 4; it is moved behind door 3.

You are then allowed to choose a different door.
This is repeated until you win the prize.
What is the minimum number of moves that guarantees you win the prize?
What happens if there are more than 4 doors?

It couldn't be probability....



Although it's tempting to assume that if the prize begins behind door 2, then there's a 50% chance of it moving to door 1 and a 50% chance of it moving to door 3, this is not the case since it isn't specified that the assistant moves the prize at random. Instead, it made more sense to represent the game as a diagram of all possible moves the prize could make after each door is chosen.



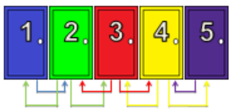
Here, the symmetry of the problem becomes apparent. If the prize begins behind an end door, the path it would take will be eliminated by choosing a door that is next to the end on the second move. Also, since a lot of the paths coincide, the only way to cut off all options is to choose at least one door twice; if the prize starts behind an odd door, it will move to even, to odd, to even after the first, second and third move respectively, similarly the opposite is true if the prize starts behind an even door.

Since we've eliminated the possibility of starting at an end door, and because of the symmetry of the case, you could begin choosing either door 2 or 3; let's say 2. The next move, as previously described, should eliminate the possibility of it starting behind an end door, and since any path leading from door 2 has now been halted, it seems only reasonable to pick door 3, behind which the prize will be if it began behind door 4.

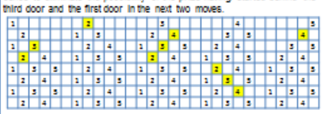
Now, the movement of the prize is solely dependent on the prize having started behind doors 1 or 3 since the other possibilities have been eradicated. Helpfully, the paths the prize would take from starting behind either of these doors have now coincided. Which means the prize will definitely be won in the next two moves, by choosing door 3 for a second time, then door 2. By this logic, this is clearly the minimum number of moves as any other step take would prolong the process of elimination. The four moves taken are

2→3→3→2 or by symmetry 3→2→2→3

If there are more than 4 doors....



The 5 door solution proved to be more challenging because now there are three doors behind which the prize could move in two directions. Again consider trying to pin down the prize by eliminating doors according to where the prize may have started. It makes sense for the first steps of the solution to be the same as the first steps to the four door solution, again, dealing with symmetry later, initially choose door 2 then door 4; if the prize was originally behind door two or door 5, it could be won by the second move. As the paths begin to overlap, it makes sense to eliminate the possibility of the prize having started behind the third door and the first door in the next two moves.



Having chosen doors 3 and 2, the possibility of the prize having started behind door 4 mean the prize may still have escaped. The above representation of the movement of the prize shows how by then choosing doors 2, 3 then 4, all possibilities are eliminated and you'll have won the prize.

Hence with 5 doors, you can guarantee a win in a minimum of 7 moves:

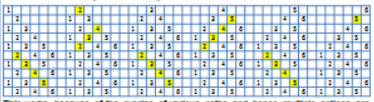
2→4→3→2→2→3→4

or by symmetry

4→2→3→4→4→3→2

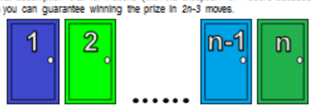
A hypothesis for even more doors...

From the previous two examples, a pattern begins to develop for the method to minimise the number of doors chosen to guarantee winning the prize. It is dependent on cutting off the paths taken by the prize which are determined by its initial position. Assuming that for any number of doors the first two moves would be the same: choosing the second door from the right then the second door from the left, a "staircase pattern" developed.



This works because of the overlap of various paths and hence multiple options are excluded when choosing one door. For 6 doors, you can guarantee winning the prize in a minimum of 9 moves.

In other work, for 7 doors, it took a minimum of 11 moves so this gave lead to a general assumption that for *n* doors (with the exception of 4 doors because of its size) you can guarantee winning the prize in 2*n*-3 moves.



The pattern of choosing the door as follows will always guarantee you win the prize:

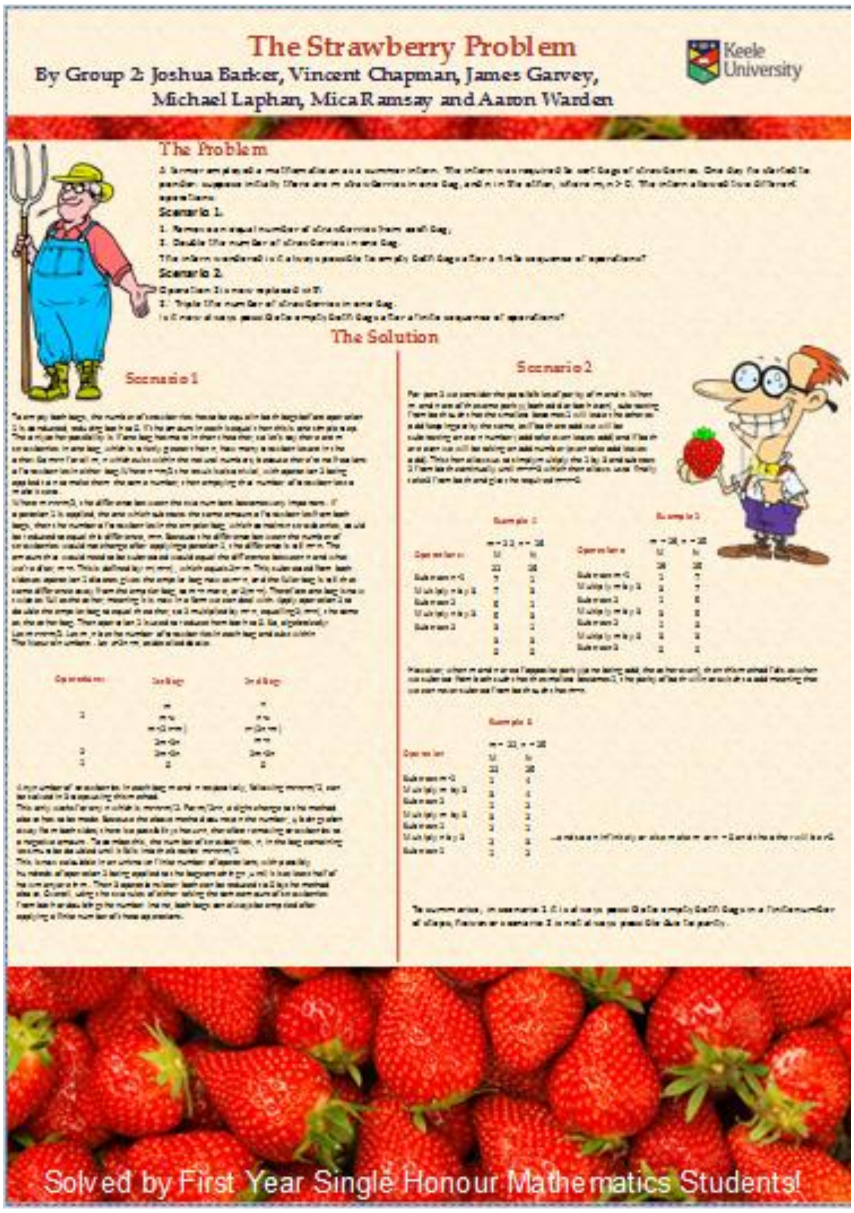
2 (n-1) (n-2)...(n/2 +1) (n/2)...3 2 2 3... (n/2) (n/2 +1)...(n-2) (n-1)

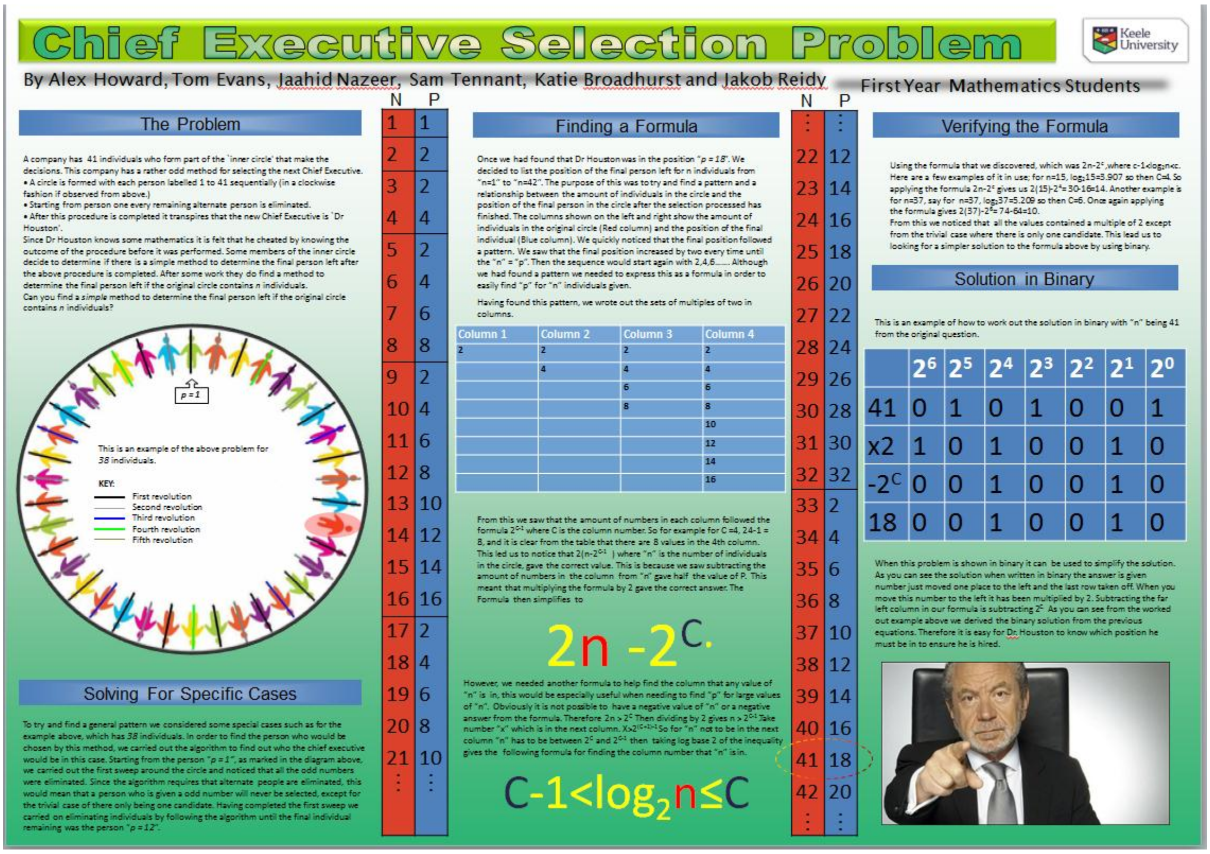
The ellipses indicate doors being chosen in consecutive succession.

This pattern does not work for 4 doors simply because there are only 2 doors from behind which the prize can move in 2 directions so the prize can be won once it has moved from behind an end door to one of the two middle doors. This is also Why the prize can be won in four moves rather than five (2n -3)

To conclude....

It took many attempts and failures in the group before concluding that this was certainly the best way to prevent leaving winning the prize to chance as it minimises the number of moves taken before the prize is won. This allows us to state that the best way to solve this and indeed any problem is to ensure you understand the problem, design a visual representation of the problem then use logic and break down the problem into its simplest form understanding how it works.


The Strawberry Problem
By Group 2: Joshua Barker, Vincent Chapman, James Garvey, Michael Laphan, Mica Ramsay and Aaron Warden
Keele University
The Problem
Scenario 1.
Scenario 2.
The Solution
Scenario 1
Scenario 2
Solved by First Year Single Honour Mathematics Students!


Chief Executive Selection Problem
Keele University
By Alex Howard, Tom Evans, Jaahid Nazeer, Sam Tennant, Katie Broadhurst and Jakob Reidy
First Year Mathematics Students
The Problem
A company has 41 individuals who form part of the 'inner circle' that make the decisions. This company has a rather odd method for selecting the next Chief Executive.
• A circle is formed with each person labelled 1 to 41 sequentially (in a clockwise fashion if observed from above.)
• Starting from person one every remaining alternate person is eliminated.
• After this procedure is completed it transpires that the new Chief Executive is 'Dr Houston'.
Since Dr Houston knows some mathematics it is felt that he cheated by knowing the outcome of the procedure before it was performed. Some members of the inner circle decide to determine if there is a simple method to determine the final person left after the above procedure is completed. After some work they do find a method to determine the final person left if the original circle contains n individuals.
Can you find a simple method to determine the final person left if the original circle contains n individuals?
p = 1
This is an example of the above problem for 38 individuals.
KEY:
First revolution
Second revolution
Third revolution
Fourth revolution
Fifth revolution
Solving For Specific Cases
To try and find a general pattern we considered some special cases such as for the example above, which has 38 individuals. In order to find the person who would be chosen by this method, we carried out the algorithm to find out who the chief executive would be in this case. Starting from the person "p = 1", as marked in the diagram above, we carried out the first sweep around the circle and noticed that all the odd numbers were eliminated. Since the algorithm requires that alternate people are eliminated, this would mean that a person who is given a odd number will never be selected, except for the trivial case of there only being one candidate. Having completed the first sweep we carried on eliminating individuals by following the algorithm until the final individual remaining was the person "p = 12".
N P
1 1
2 2
3 2
4 4
5 2
6 4
7 6
8 8
9 2
10 4
11 6
12 8
13 10
14 12
15 14
16 16
17 2
18 4
19 6
20 8
21 10
⋮ ⋮
Finding a Formula
Once we had found that Dr Houston was in the position "p = 18". We decided to list the position of the final person left for n individuals from "n=1" to "n=42". The purpose of this was to try and find a pattern and a relationship between the amount of individuals in the circle and the position of the final person in the circle after the selection processed has finished. The columns shown on the left and right show the amount of individuals in the original circle (Red column) and the position of the final individual (Blue column). We quickly noticed that the final position followed a pattern. We saw that the final position increased by two every time until the "n" = "p". Then the sequence would start again with 2,4,6...... Although we had found a pattern we needed to express this as a formula in order to easily find "p" for "n" individuals given.
Having found this pattern, we wrote out the sets of multiples of two in columns.
Column 1 Column 2 Column 3 Column 4
2 2 2 2
4 4 4
6 6
8 8
10
12
14
16
From this we saw that the amount of numbers in each column followed the formula 2^(C-1) where C is the column number. So for example for C =4, 24-1 = 8, and it is clear from the table that there are 8 values in the 4th column. This led us to notice that 2(n-2^(C-1)) where "n" is the number of individuals in the circle, gave the correct value. This is because we saw subtracting the amount of numbers in the column from "n" gave half the value of P. This meant that multiplying the formula by 2 gave the correct answer. The Formula then simplifies to
2n -2^C.
However, we needed another formula to help find the column that any value of "n" is in, this would be especially useful when needing to find "p" for large values of "n". Obviously it is not possible to have a negative value of "n" or a negative answer from the formula. Therefore 2n > 2^C Then dividing by 2 gives n > 2^(C-1) Take number "x" which is in the next column. X>2^((C+1)-1) So for "n" not to be in the next column "n" has to be between 2^C and 2^(C-1) then taking log base 2 of the inequality gives the following formula for finding the column number that "n" is in.
C-1<log₂n≤C
N P
⋮ ⋮
22 12
23 14
24 16
25 18
26 20
27 22
28 24
29 26
30 28
31 30
32 32
33 2
34 4
35 6
36 8
37 10
38 12
39 14
40 16
41 18
42 20
⋮ ⋮
Verifying the Formula
Using the formula that we discovered, which was 2n-2^c ,where c-1<log₂n<c. Here are a few examples of it in use; for n=15, log₂15=3.907 so then C=4. So applying the formula 2n-2^c gives us 2(15)-2^4= 30-16=14. Another example is for n=37, say for n=37, log₂37=5.209 so then C=6. Once again applying the formula gives 2(37)-2^6= 74-64=10.
From this we noticed that all the values contained a multiple of 2 except from the trivial case where there is only one candidate. This lead us to looking for a simpler solution to the formula above by using binary.
Solution in Binary
This is an example of how to work out the solution in binary with "n" being 41 from the original question.
2^6 2^5 2^4 2^3 2^2 2^1 2^0
41 0 1 0 1 0 0 1
x2 1 0 1 0 0 1 0
-2^C 0 0 1 0 0 1 0
18 0 0 1 0 0 1 0
When this problem is shown in binary it can be used to simplify the solution. As you can see the solution when written in binary the answer is given number just moved one place to the left and the last row taken off. When you move this number to the left it has been multiplied by 2. Subtracting the far left column in our formula is subtracting 2^C. As you can see from the worked out example above we derived the binary solution from the previous equations. Therefore it is easy for Dr. Houston to know which position he must be in to ensure he is hired.

Appendix 5

The following are examples of the types of problems used during the module at various stages. As noted in Appendix 2 we used questions from the UKMT Senior Team Mathematics Challenge. These questions are available freely on the web and we have provided links to these questions.

Starter group problems

You have accidentally left the plug out of the bath and are attempting to fill the bath with both taps at maximum. The hot tap takes 5 minutes to fill the bath and the cold tap 4 minutes. The water completely drains from a full bath in 20 minutes.
In how many minutes will the bath be filled?

Group Project 1

Problem 1: Find all (positive or negative) integers n for which $n^2 + 20n + 11$ is a perfect square. Remember that you must justify that you have found them all.

Problem 2: A company has 41 individuals who form part of the `inner circle' that make the decisions. This company has a rather odd method for selecting the next Chief Executive.

- A circle is formed with each person labelled 1 to 41 sequentially (in a clockwise fashion if observed from above.)
- Starting from person one every remaining alternate person is eliminated.
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Since Dr Houston knows some mathematics it is felt that he cheated by knowing the outcome of the procedure before it was performed. Some members of the inner circle decide to determine if there is a simple method to determine the final person left after the above procedure is completed. After some work they do find a method to determine the final person left if the original circle contains n individuals.

Can you find a simple method to determine the final person left if the original circle contains n individuals? You should make sure you explain why the method works.

Problem 3: You are taking part in a game show. There are four doors and initially a prize is placed behind one of the doors, nothing is placed behind the other doors.

The procedure is as follows:

- You select a door,
- The door is opened, if the prize is behind the door then you win,
- If there is nothing behind the door then the door is closed and the host's assistance does the following:
    - If the prize is behind door 1, then it is moved to behind door 2;
    - If the prize is behind door 2, then it is either moved to behind door 1 or behind door 3;
    - If the prize is behind door 3, then it is either moved to behind door 2 or behind door 4;
    - If the prize is behind door 4, then it is moved to behind door 3.
- Once the prize has been moved then you are allowed to choose a door again.

Can you find a strategy that minimises the number of choices you must make before you can guarantee you will win the prize? What happens if there are more than four doors?

Group Project 2

Sergey Bubka currently holds the pole vault world record at 6.14metres. This record was set on 31 July 1994 in Italy. Only seventeen men have ever exceeded 6 metres and experts generally believe that the best that could ever be achieved is 6.4 metres.
Develop a simple model to explain potential heights that an athlete could reach.

What percentage of the world's water is contained in a cow?

Group Project 3

This project is described in Appendix 2.