Pedagogic practices and interweaving narratives in AS Mathematics classrooms

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Prologue

As you set out to read this paper you may wonder about the story it will tell. Already as authors we are manipulating your engagement as a (potential) reader. Hopefully, our choice of title appeals suggesting that our focus will be AS mathematics lessons and that we will be viewing these by taking account of the practices that teachers employ within their teaching and the different stories that are involved. The narrative device of speaking to you directly is perhaps unusual in an academic paper: you may be wondering how else we will attempt to maintain your interest and engagement and how we will sequence events to ensure that the points we wish to make potentially have maximum impact. We raise these issues at the outset, as narrators ourselves, to help us suggest that mathematics teachers in their design and implementation of lessons make similar choices: they narrate a story, or stories, for their students where the development of mathematics is in the main central, but may not always be so. However, their medium, the lesson, has much more potential than the page available to us here. They consequently have other choices to make about how to engage the audience, their class, in their unfolding narrative(s): what pedagogic practices will they use to maximise impact, entertain, and so on? In this way teachers, therefore, might be considered as directors of a performance art production, who may at times decide to incorporate the audience to a greater or lesser extent, in the story they tell.

Introduction

The ESRC (Economic and Social Research Council) funded research project, 'Keeping open the door to mathematically demanding courses in Further and Higher Education' involves both case study research investigating classroom cultures and pedagogic practices and individual students' narratives of identity together with quantitative analysis of measures of value added to learning outcomes in an attempt to investigate the effectiveness of two different programmes of AS mathematics for post-16 students. Here we focus on the classroom experiences of students drawing on data that has been collected in the ethnographic tradition with video and audio recordings, photographs and researcher notes together with follow-up interviews with students both in small groups and individually and pre- and post- lesson interviews with the teachers involved.

The two programmes are for post-compulsory study of mathematics in the early stages of preparation for university entry. The most common route is to follow the GCE A Level Mathematics course (with the first year of study culminating in an initial subsidiary AS qualification) which has a long tradition as the academic path potentially leading to a wide range of university courses, particularly in mathematics, the sciences and technology. A new programme of study "Use of Mathematics" leads only to AS Level and is designed to focus on modeling and applications of mathematics likely to be of assistance to students either in their current or future studies, for example, in sciences, engineering, economics and so on. This new course is designed to overcome the traditional A level barrier by appealing to students with relatively limited prior attainment in mathematics. By engaging with computer technology, formative assessment by portfolio and through its emphasis on inquiry, communication and relating mathematics to its applications it is designed to include and keep engaged more students in mathematics for longer.

Lessons as pedagogic events

In this paper we draw on classroom observation data from two of our five case study colleges and explore how we might come to understand the impact that lessons as pedagogic events have on students. We suggest that these experiences within mathematics lessons are crucial to students in their development (whether positive or negative) of their identities as learners of mathematics and consequently their ability to pursue mathematically demanding subjects. Our observations reflect the findings of Boaler & Greeno (2000) that learning environments are crucial in determining the development of students' development of different cultural models and emerging identities as learners of mathematics (Davis et al, 2007). In contrasting "didactic", with emphasis on transmissions of rules and procedures with more "sociable" approaches where exploration and connection-making is encouraged, it is suggested that the latter can significantly encourage some students to more positively engage with mathematics. Whilst recognising that different pedagogic practices appear dominant in setting a 'tone' for the classrooms of different teachers and clearly impact in a major way on students, we suggest here that perhaps we should look beyond this at another subtle aspect of the totality of students' mathematical experiences and to do so turn to the construct of narrative. However, before exploring this in a little more detail we firstly examine what we mean by "pedagogic practices". These we take to be the practices which the teachers employ to engage, however loosely, their students with subject matter: resulting from these practices are activities for both the teacher and students. These resulting activities may be more or less teacher- or student- centered in that they may require more or less input from one party or the other with either the teacher or students or both working individually in groups or together. For example "practices" could include, amongst others:

- monological transmission of information by the teacher to the whole class (this being very much teacher-centered with students maybe taking notes or listening relatively passively);
- group- discussion followed by whole-class discussion (with the teacher adopting a different role in each phase. Perhaps shifting from facilitator to summariser and individual students contributing in different ways in the two different situations);
- students "playing a game" (with the teacher facilitating this by setting the rules and time-frame for students to be active participants and intellectually engaged);

…and so on.

Whilst some of these practices are perhaps better suited to lessons in mathematics than in other subjects they might none-the-less be employed by teachers of other disciplines. In recent years building on the work of Swan (2006) certain practices have been promoted for the teaching of mathematics in particular and curriculum resources for post-16 students have been developed to support these practices (DfES, 2005). These include:

- classifying mathematical objects
- interpreting multiple representations (for example, graphical and algebraic representations of functions)

- evaluating the validity of statements and generalisations
- analyzing reasoning and solutions
- correcting and diagnosing common mistakes
- resolving problems that generate cognitive conflict.

We have observed such practices at certain times in almost all of our case study colleges, indeed you may recognise characteristics of them in the accounts we give of two particular lessons below. However, whilst both teachers, and indeed students, see these as being important in engaging students in mathematics we propose that there is something of importance beyond this engagement that crucially involves the development of the mathematics itself. Other studies, particularly international studies such as TIMSS have using a large scale video sample analysed aspects of mathematics lessons in an attempt to characterise typical lesson structures across different countries. Stigler and Hiebert (1999) suggest that mathematics lessons are culturally and historically bound and consequently within a particular country will tend to develop a common format that is difficult to alter. Teachers share "scripts" to which they plan their teaching: these become culturally bound over time with teachers having early experiences of the common patterns of lessons when they themselves are pupils. The TIMSS video study draws on a substantial number of videos of mathematics lessons across seven countries which were analysed to examine

i. the context of the lesson;
ii. the organization and structure of the lesson environment;
iii. the kind of mathematics studied;
iv. the way in which the mathematics was studied.

Quantifying different aspects under each of these broad headings allowed the study to identify commonality in the way teachers operate within a country. Here, however, the lessons we present are deliberately chosen to exemplify *different* practices, and consequently do not necessarily appear to conform to a national stereo-type of lesson structure, although Lesson 1 perhaps reflects more closely what might be taken as the "normative" cultural script (Wierzbicka, 1999). Whilst it is difficult to find quantitative evidence of typical practices in post-16 mathematics lessons, our own experience is that many lessons consist of vast tracts dominated by teacher exposition of rules and procedures punctuated by periods in which students, working individually, practise these.. The most recent Ofsted report (Ofsted, 2006) that tackles this issue attempts to encourage a shift to the more student-centered practices described above but does hint at this common pattern:

> "… in promoting a really secure understanding of mathematical ideas, in stimulating students to think for themselves and to apply their knowledge and skills in unfamiliar situations, the picture was less encouraging. In approximately half of the lessons observed, the teaching did not sufficiently encourage these important aspects of learning in mathematics. ….. In contrast, teaching which presented mathematics as a collection of arbitrary rules and provided a narrow range of learning activities did not motivate students and limited their achievement." (p.4)

Our concern, therefore, is to explore in greater depth students' potential engagement with the mathematics as it develops and ultimately gain some understanding of the impact this might have on their likelihood of continued engagement in its study. In an attempt to do this we explore the mathematical development through the lens of "narrative", in the sense of Ricouer, and as developed in educational settings by Bruner (1996) and others. We conceptualise the teacher as "narrator" revealing a mathematical plot whilst drawing on a range of pedagogic practices in an attempt to engage his or her audience in different ways. For example, as we have suggested above, the students may be actively engaged in working together matching different representations of the same mathematical object, with the teacher seemingly taking a back-seat role, whilst at a later stage, or on another occasion, the same teacher may be much more to the foreground modelling the solution to an examination question with the students taking notes. Here, however, our concern is to focus not only on the method of engagement chosen but also the structuring of the mathematics; in other words, the story that the teacher tells about this. At issue then are the teacher's understanding and beliefs, not only about the form of mathematics lessons that best prepare students in the subject but also their knowledge of, and beliefs about, the mathematical content and processes involved and how these are best ordered and revealed to students. As Shulman (1986) suggested there is a danger when considering the practice of teachers, and therefore presumably the experience of learners, of focussing exclusively on pedagogic practices. In his seminal paper of 1986 he proposed that we should additionally take account of the knowledge that teachers use in presenting subject content to students, which he termed pedagogic content knowledge (PCK). PCK takes account of the fact that teaching requires more than knowledge of subject alone but also requires knowledge about different ways of representing knowledge to learners, how knowledge interconnects within and across topics, likely ways in which students will make sense of what they are asked to learn including knowledge of likely misconceptions, use of models and other devices, and so on. In critiquing Shulman's point of view, Pieteg (1997)_ argued that in fact there is no distinction in principle between the 'subject of mathematic' and 'pedagogy for mathematics'- he argued that essentially every mathematical argument, and so every mathematical text, has a 'pedagogic' function vis-à-vis the intended audience. Indeed Bernstein (1996) has also argued that texts and discourses must be increasingly recognisable as pedagogic. Our point then builds on this, and suggests that this pedagogy must have narrative, and therefore that any effective mathematical pedagogy or argument must have a genre of narrative. Our aim though is to find out what narratives there actually are out there!

Teachers as narrators of mathematics

Here, therefore, in an attempt to understand, how teachers structure mathematics for their students we consider the teacher as storyteller weaving together episodes in the development of a mathematical argument with each episode contributing to a significant plot representing something greater than the sum of the individual episodes. How the teacher weaves together *her* story about the mathematics at issue, we suggest, reflects her epistemological and pedagogical beliefs. Here we propose that this narrative has a significant role to play in determining the cultural models of mathematics that lessons in colleges offer and therefore with which students may be more or less inclined to engage.

In their discussion of mathematical narrative Mor and Noss (in press) suggest that it is when teachers make the mathematics accessible that narrative results; they exemplify this by suggesting that it is the attempt to humanise the propositional statement 2 + 2 = 4 in the form "*If you had two marbles, and I gave you two more, you would have four*" that leads to narrative. Notice that even in this brief sentence key features of narrative emerge: for example, temporality i.e. a sense of time with a starting position, and an event that leads to a conclusion or end point. We suggest that in communicating mathematics to others, or even in making sense of it oneself, it is necessary to develop a narrative about the mathematics itself and which may incorporate other strands. Further, we propose that with the lesson, or sequence of lessons, as our unit of analysis we can distinguish two strands of narrative: (i) a mathematical strand which is distinct and different from (ii) a 'social' strand that may incorporate details of context and attempts by the teacher to humanise the mathematics and engage students with it by their use of discourse and choice of pedagogic practices. The mathematical strand of the resulting overall narrative of a lesson we suggest results from the way in which individual teachers organize and unfold *their* development of a mathematical argument even when this is considered extracted and isolated from attempts to use motivating contexts.

To illustrate the idea of mathematical narrative consider how two different teachers might attempt to introduce preliminary ideas of differentiation in elementary calculus (this will be appropriate to the first lesson we recount below).

i. One teacher might develop a "traditional" approach firstly considering the gradient between two points on a function (for example, between $x = 2$ and $x = 3$ for the function $f(x) = x^2$) and investigating the nature of this as the two points become increasingly close (for example between $x = 2$ and $x = 2.5$, then between $x = 2$ and $x = 2.25$ and so on effectively zooming in on the gradient at $x = 2$). Following this the teacher may algebraically derive the gradient at a point for a function (often $f(x) = x^2$) "from first principles". As the algebra required to work with functions other than the simplest powers of x is often too difficult for students the teacher then often has to rely on "telling" students the general result that the differential of x^n is nx^{n-1}.

ii. Contrast this with a possible approach of a different teacher who in his narrative reflects the ideas of "graphic calculus" as advocated by David Tall (see for example, Tall and West (1986))). In this case, using technology the teacher is able to direct students' attention to "local straightness" by zooming in at a point on a curve. This suggests that at a point the gradient of a curve can be found by calculating the gradient of a very short line segment. Indeed computer software can be used to calculate such gradients at successive neighbouring points on a curve allowing a "gradient function" to be plotted. This can be investigated dynamically using technology and appropriate software. By inspection, and using their prior knowledge of functions, perhaps drawing on ideas of geometrical transformations of functions, students may then deduce the algebraic form of gradient functions for a range of functions familiar to them, and check using the graph plotting facilities of the software. Because the power of the technology allows them to work quickly they can often deduce for themselves simple rules of differentiation for a range of functions.

We suggest that these are indeed two different mathematical narratives resulting from which students will have different ways of seeing, and understanding the mathematics concerned. Importantly we hypothesise that when this mathematical narrative is brought together with the teacher's chosen pedagogic practices and social discourse the result will be a narrative construction that will be unique to an individual lesson. This will, therefore, affect the students' engagement with mathematics in complex and subtle ways which will ultimately have an important bearing on their emerging identities as learners and users of mathematics.

It is worth noting that Cooper (1998) and Cooper & Dunne's work (1998 and 2000) might be re-conceptualised in this light: the assessment tasks that they critique as disadvantaging working class learners are precisely those whose story gets misread, or inappropriately culturally-scripted by the learner. In a teacher's authoring of their lessons which we see as scripts or narratives, we may perhaps detect such miscommunications between the teacher and the learner in how they are to be understood/encoded.

In justification of our claim that teachers use a discourse of mathematical narrative in their classrooms we explore here some of the key features of narrative in general, as identified by Bruner, (1996) before exemplifying this thinking in relation to two different lessons. Firstly, perhaps the most obvious and most easily identifiable of the universals of narrative is how the narrator unfolds crucial events in a temporal sequence, with a beginning (a problem to be solved), middle and end (resolution of this). Of course, in the case of a novel, the author may not reveal events in the temporal sequence in which they actually (or as they are imagined by the author to have) occurred; good story-tellers may make use of flash-backs and flash-forwards, introducing characters and events to engage and surprise the reader. We suggest that the teacher as narrator makes similar choices about the building blocks of the mathematics they seek to introduce: they too even have the possibility of using narrative devices such as flash-forwards, perhaps using a glimpse of the end-point to act as an advance-organiser or motivator for their class.

Perhaps on occasions a lesson will be too small a unit of analysis to fully follow the developing story of a mathematical topic as a teacher unfolds this with his or her class; however, over time, perhaps within, and beyond, the lesson, the teacher reveals a sequence of episodes of *their* mathematical narrative. The example of the two ways in which teachers might introduce differentiation, outlined above, are useful in exemplifying very different approaches with the teachers selecting different "mathematical characters" (for example, the "gradient function" or "differentiation from first principles") and organising a sequencing of the introduction of these to cover the same topic. Given any topic in mathematics it seems that individual teachers will choose different "mathematical characters" that they will introduce and different ways in which to sequence this. Although many may follow the departmental scheme of work or class text book, even then they often make decisions about the order in which to cover sections of a chapter, what to miss out, what to emphasise, and so on.

A further feature of narrative is its hermeneutic composition: how the episodes of a narrative have meaning on their own, but how in combination they provide greater meaning than their individual parts and, in retrospect, how the whole narrative adds meaning to the individual constituent episodes. This appears to be an important feature of mathematical narrative where, in coming to understand a particular concept, one needs to draw on concepts and ideas met earlier and often across different branches of mathematics. For example, consider the first suggested narrative associated with learning differentiation above, the narrator asks the audience to draw on understanding of functions and graphs, the calculation of the gradient of straight line segments, the development of the idea of limit and the associated calculation that gives gradient at a point, followed by the development of an algebraic argument that leads to a general expression for gradient – the differential. The resultant concept of derivative is therefore based on the combination of these different mathematical ideas, each of which has meaning and importance in its own right and which is enhanced by the result of the overall narrative: the concept of derivative itself. This feature of mathematical narrative is clearly exemplified by lesson 2 below.

For the teacher who has carefully crafted his or her tale, the episodes they introduce reflect their understanding of what has gone before and moves the plot forward, but mathematically do the learners see the wood for the trees? Do they get the story?

To be worth telling, Bruner further suggests that narratives in general should run counter to expectancy and have "trouble" as a central feature. In mathematical narrative, particularly when the narrator wants to engage the audience in a dialogic pedagogy such a feature, a 'problematic' is essential, provoking different points of view from a shared understanding of an initial situation (Ryan & Williams, 2007). The very fact that students may have different points of view when exploring a problematic, will result, at least for some, in deviation from the expected. Presumably the fact that, in mathematics classrooms, teachers are often confronted with common misconceptions, their narratives, as interpreted by their students, are often responsible for introducing elements that run contrary to expectations and even, for at least some, surprise. Indeed, many mathematics educators suggest that teachers should plan an 'element of surprise' into their lessons (for example see, Movshovitz-Hadar, 1988).

Finally, how do we make sense of all the different narratives with which we engage? As is often the case, we seek order and pattern from variety: as Bruner suggests in the case of narrative in books, films etc. in attempting to impose order on all of the particular narratives with which we engage different genres can be considered to emerge. In the case of novels we recognise not just genres such as romance, crime, westerns and so on, within these we might identify commonality such as "the good who fall from grace", "the loser who eventually strikes it lucky" and so on.

We suspect that we will equally be able to identify different genres of narrative form in mathematics lessons, and that these may very much reflect the teacher's deep seated pedagogic beliefs. This is likely to lead individual teachers to develop and often use a particular narrative style or genre much in the way that the work of a novelist can often be similarly categorised. In seeking to investigate two different programmes of AS mathematics, one of which is designed to support modelling and applications whereas the other does not, we might expect that the role of application and context may lead to significant differences in the mathematical narrative forms of teachers working on the two programmes. For example, we might expect some teachers to develop a genre of mathematical narrative in which the learning of mathematics is often motivated from problems developed in context. This may reflect the way in which Freudenthal (1983) and followers (for example, Treffers (1993) and Van den Heuvel-Panhuizen (2001)) attempt to motivate the learning of mathematics by stimulating students to develop models *of* contextual situations, (which they describe as horizontal mathematisation) leading to the development of mathematics *for* (in the sense of vertical mathematisation) the construction of new mathematics itself. Lesson 2 below describes a teacher using such a genre of mathematical narrative. However, other teachers, as we have observed in lessons not reported here, might use a genre of mathematical narrative in which firstly the mathematics in its more abstract form is generated and developed after which this is applied to solve contextual problems. It is important to emphasise that teachers may choose to use a range of different pedagogic practices whilst developing their narrative about a topic using either of these distinctively different genres.

These key features of narrative form appear, therefore, to be identifiable in the way that teachers unfold mathematics for their students: thus we are led to propose that mathematics itself is developed as a distinct narrative by teachers in their classrooms with the plot devised by the teacher reflecting their understanding of their own interpretation of individual and collections of mathematical stories. The classroom events teacher organise for their students have as a central feature the teacher's (re-) interpretation of stories that were told to them together with their own thinking about, and understanding of, mathematical topics and their interconnections.

However, as our accounts of lessons illustrate, in mathematics lessons teachers, in their discourse with students, do not entirely focus on the mathematics. Crucial also are the social interactions between teachers and students: we detect that whilst at times these may be spontaneous they are also often planned to interweave and interplay with the developing mathematical narrative in such a way that their social narrative may, or may not, add power to and strengthen this. On occasions we detect teachers using 'social' narrative in a way that has the potential to be both engaging and add meaning to the mathematics being developed.

Thus in an attempt to analyse the lessons as pedagogic events we are led to consider using a two-dimensional framework (Figure 1): (i) with one dimension taking into account the teacher's narrative about the mathematics itself; (ii) with a second dimension which is socially focused that takes account of the pedagogic practices that the teacher uses and which reflect the culture of mathematics teaching and learning within which the teacher works, together with a social discourse. The students' experience therefore reflects the intersection of these two dimensions: even if two teachers were to have the same mathematical narrative it is unlikely that their social narrative and choice of pedagogic practices would be the same and therefore students in their two classes would have different experiences even before the crucial issue of their interpretation of these experiences is taken into account.

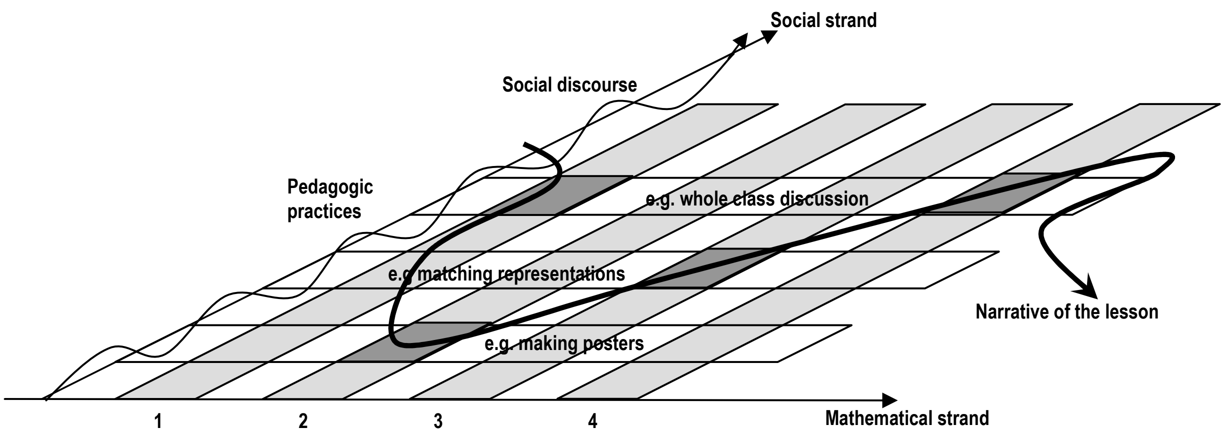


Figure 1: Schema illustrating two dimensional framework used to analyse the narrative of mathematics lessons

We now turn to the first of three lessons from one of our case study colleges which we will recount in some detail. At this point it should be borne in mind that although we are working with video and audio data the story we tell here of this lesson is *our* story as researchers and although we do not until the end of its presentation here comment in any detail our selection of what to recount is due to judicious selection of data relating to the lesson and as such already reflects the framework we propose above.

Lesson 1: "The worms"

This lesson is in the first term of the AS "Pure" course in a college in the North of England: it focuses on applications of differentiation. The teacher in an interview some time before the lesson observation told us:

> "My teaching style has always been quite traditional. Pretty much a typical lesson of mine will be aim and objectives on the board or recap what we did last time, an introduction of what we're doing this time and I expect students to make notes because I don't just… literally only use the textbook for questions, I often teach a topic differently than in the textbook and then students try some… sometimes I'll do activities lab we saw [teacher] doing yesterday."

This summary is borne out by the lessons we observed, and the one we describe in some detail here, the transcript of which gives stark evidence of a monological transmission style with very few utterances by anyone other than the teacher.

In the introductory phase of the lesson the teacher initially drew attention to differentiation as the abstract idea of rate of change of y with respect to x referring to the notation $\frac{dy}{dx}$. To illustrate that this might be applied to a "real" situation and that different variables other than x and y might be involved the teacher suggests that at issue might be the rate of change of velocity with respect to time, and went on to ask if anyone knows the "special name" for this particular rate of change as the only question in the first ten minutes of the lesson.

Perhaps the transmission style the teacher employed is encapsulated in his statement at this point of the lesson that, "we just need a couple of definitions before we can move on to what I wanted to look at in detail today."

He drew a non-specific/general curved line and emphasised that the gradient at a specific point is given by the differential of the function introducing appropriate notation $f(x)$ and $f'(x)$. At this point he introduced the "new stuff" – the average gradient, or gradient of the chord, between two points (A and B) on the curve, although he did suggest that this idea had been met by the group when differentiation was first introduced. Again the transmission and teacher-centric style employed is evident at this point as he stated that, "what I'm about to say now are the two most important things you need to learn this lesson." Here he re-emphasised that the gradient at a point is found using differentiation, $\frac{dy}{dx}$, and average gradient between two points is found using the gradient of a chord, and he introduced the notation $\frac{\delta y}{\delta x}$.

Following this the lesson moved to a second phase in which the teacher modelled how to answer a problem of a form that they would practise in the final stage of the lesson. However, at this point of the lesson the teacher introduced a 'social' strand of narrative that from this point interweaves with the mathematical narrative. This revolved around an imagined world in which the teacher builds a problem situation around the worms in his garden: as you will see this is not a 'real' context but perhaps is 'realisable' as in the RME approach (see for example, Van den Heuvel-Panhuizen (2001)). As this extract of a transcript of the lesson demonstrates this strand of the teacher's narrative with which he expects the students to engage is not insubstantial.

> "So I went into my garden, true story this, and I started digging up some worms. Alright? So I took my fork and I dug up lots of worms and they were all of different

sizes so that's quite interesting in the first place. So I thought, well, I wonder if there's any relation between the age of these worms and their length so I collected as many worms as I had time for. For the visual learners amongst you, here's one of them. This is, in fact, it's Japanese. Could be German. Who knows? Ok, this is one of the worms I collected. So I collected till I'd got enough, a decent sample size, right? And I measured these worms, how long they were. I then asked them how old they were. They were quite co-operative. And I plotted how long the worms were at particular ages and, to my surprise, and remember you're not making notes, this is background, to my surprise, when I plotted the age of the worm to its length, all the points roughly lay on what looks to me like a quadratic so, of course, as you yourselves, I got quite excited at that and I thought, well, if I could find the equation of that quadratic, I'm quids in, yeah? I could predict the length of worms at different ages that I didn't have so I got very excited. I also noticed that when the worm wasn't born its age was zero so that was spot on, that fits nicely, so I do know one point that lies on this potential quadratic, quadratic with a negative coefficient of the squared term."

With brief reference to techniques that students will have met at GCSE of fitting a quadratic curve to model data such as this the teacher went on to introduce the equation $l = 8t - \frac{1}{2}(t^2)$ that he teacher claimed to have found for a curve that fits his imaginary data of worm length, l millimeters, at time, t years. The first part of the problem that he set was to find the rate of growth of these worms in the first year of their lives. He immediately translated this applied problem for the students into the more abstract mathematical form of having to calculate "delta l" by "delta t". He then proceeded to demonstrate how to find the average gradient by firstly finding l when $t = 01$ and then proceeding to calculate the increase in l $\left(7\frac{1}{2} - 0\right)$ divided by the increase in t (ie 1). After brief comment by the teacher that a rate of growth of $7\frac{1}{2}$ millimetres per year was, "Quite a lot really" the teacher repeated the procedure carrying out all of the calculations at each stage to find the average rate of growth during the fourth year. In conclusion of this phase of the lesson the teacher "discussed", by asking questions that he answered, the validity of the answers he had found so far:

> "Has the result surprised you or not? 7.5 millimeters per year in the first year, 4.5 millimeters per year in year 3 to year 4. Does that make sense that a worm grows really quickly at first and then starts slowing down its growth rate? That seems sensible to me, I think we do the same. Obviously, I'm still growing but…ok."

In what may, due to the shift in the mathematics involved, be considered a third phase of the lesson the teacher posed the question,

> "What is the rate of growth after 3 years? Not what is the average rate of growth. Exactly, on the worm's third birthday - at that instant, what is its rate of growth?"

Again in this phase the teacher modelled how to find an answer by differentiating the function $l = 8t - \frac{1}{2}(t^2)$ and substituting $t = 3$ to give a rate of change of 5 millimetres per year. Again the teacher asked the class to consider the likely validity of this answer by comparing it with the average rates of change he had found for the first and third years of the worm's life.

In a fourth phase of the lesson the teacher posed the question, "How many years before the worm is fully grown?" After suggesting that the students should think about this in terms of the rate of change of the length of the worm a student made the second intervention of the lesson suggesting that this is at a stationary point. Re-interpreting this, the teacher pointed out,

> "In other words, the gradient is zero. When the worm is now fully grown, it's no longer growing so the rate of change of the length with respect to time is zero."

He proceeded to demonstrate that in this case $t = 8$, and once again considered this in the light of the context of the situation, re-introducing a social element of narrative:

> "Now, I hope there aren't any biologists here who are going to tell me that worms don't live to 8 years old. They do in my garden, they wouldn't lie to me. I asked them and they said they were honest about their age. So 8 years is when they stop growing. They're not dead. They just stop growing."

The teacher moved on to consider when his model is not valid. However, for the first time in this lesson he shifted away from assuming total responsibility for answering the question, a shift that he signalled clearly to the class:

> "There's one more question, a rather interesting one to think about, which I'll open to the floor to think about. That is the last question which says, are there any values of t for which the formula is not valid and also explain your answer? It's an open question to anybody. Yeah?"

A student suggested that the model is not valid for values of t less than zero and bigger than sixteen because for these values the formula gives values of l less than zero. In an attempt to funnel students towards a better or more accurate answer the teacher asked the group to consider what happens when t is greater than eight and one of them suggested that in this case the worms would be shrinking. Again the teacher returned to his social narrative:

> "Because I know older humans, when they get older, they sort of walk around like this but they haven't really shrunk, have they? They've just started to walk around like that but they haven't shrunk. And we can't have that for worms either. So it's not valid. So does that mean my formula, all my hard work, was a waste of time? No. It only means that the worms I found in my back garden, I didn't find any older than 8. You see? So this formula, based on my experiment in my back garden, was only valid for worms up to 8 years old because that's all I found. You can't assume that for more than 8 years, it's still going to follow the same pattern as the information I collected. It might be a totally different formula for worms older than 8, ok? So this formula is only valid for worms up until the age of 8."

In a penultimate phase of the lesson the teacher set the class some practice questions of a similar nature after briefly summarizing what he had done and emphasising the difference between finding the rate of change at an instant and finding the average rate of change between two points.

Finally, the teacher set the scene for the next lesson, describing the types of questions they would be expected to apply differentiation techniques to solve. Here, once again, he used his imaginary worms to in an attempt to motivate and/or 'socially' make connections with this lesson:

> "I was rather loath to…I got quite attached to these worms with measuring them and what have you so I was a bit reluctant to release them into the wild so next lesson, what we're going to do is…I rummaged around my shed and I found a bit of rope. I wasn't going to buy a new one. Piece of fencing really and I decided that in my garden I want to make a little enclosure for my worm sanctuary. I've only got a fixed amount of fencing that I found, ok? So what I want to do…you don't have to write anything down, we'll go over this next time. My backyard, I've got a wall there and I've got a wall there. Now, if I take this fencing to make a rectangle, which I'd like them to have a little rectangle to live in, I could make a really narrow rectangle like that or I could make one like that but I am restricted by the amount of wiring I've got. So is that differentiation? I think it is because with that constraint, the amount of wiring I've got, I'm seeking to maximize the area. I want to find out which rectangle will have the most area because I want my worms to be free range worms. I don't want them to be like battery hens, battery worms, do I? So I want the biggest area possible for my worms, with the restrictions of how much fencing I have in my shed. Ok? That's what we'll do in the first lesson on Friday. In the 2nd lesson on Friday, I've decided I've become really attached to these worms and I want to take them on holiday with me so in order to take them on holiday with me, I want to make a box for them to take them on the plane. Right? Now, I've rummaged around in my shed and I've found a piece of tin about this size. So if I fold up, if I cut off the corners and fold up the sides, I could make a little box for my worms. But should I make a small box that's really tall and all my worms will be cramped together, like sardines, or I could make a flat box like that with a little bit of room to crawl around in or whatever it is they do. So, therefore, is that differentiation? Yes, I am seeking, with a restriction of how much metal I've had, to maximize the volume for my worms so I can take them on holiday. Ok? So that's what we're doing on Friday. I might even bring my worms in on Friday."

Lesson 1 analysis: Pedagogic practices and narrative strands

In our description of the lesson above we have suggested that it can be considered as consisting of a number of distinct phases. In our analysis of lessons different phases are often discernable due to a change of pedagogic practice employed by the teacher, and therefore the different modes of engagement of teacher and students. For example, in the penultimate phase of this lesson the students practise their application of techniques whereas in the final phase they hear the "trailer" for the forthcoming lesson. However, at other times different phases are suggested by a distinctive shift in mathematics: for example, early in this lesson students have to shift their thinking from considering how to find average gradient to consider the behaviour of a quadratic function. We suggest, therefore, that different phases of lessons can be identified by reference to changes in pedagogic practices or introduction of a new chapter in either mathematical or 'social' narrative.

This particular lesson has relatively little variation in pedagogic practice, with the teacher in the main choosing to use a predominant transmission style for long stretches with the only interruption being a substantial period in which students practised the techniques that he had modelled to solve a range of similar problems. This resulted in a long period of passive activity for the students followed by a period in which they were more actively engaged but on the whole working individually: the result was little or no sociability in this lesson. However, the teacher does demonstrate the use of the two distinct strands of mathematical and "social" narrative and interweaves these: at times using the 'social' narrative to motivate and at other times ensuring it intersects relatively closely with the mathematics in such a way that engagement with the social requires engagement with the mathematical. For example, consider how the teacher's social and mathematical narratives are closely aligned as he discusses how to find when a worm is fully grown with students thinking about growth mathematically (considering the maximum point of the quadratic function) and socially (the teacher emphasizes that in this context the function would suggest that the worms would be shrinking and that "we can't have that for worms".)

We consider the mathematical narrative of this lesson to be of the genre "abstract skills and techniques have use in solving contextual problems" with the teacher's narrative comprising of a number of episodes that in the main build on previously met skills, techniques and understanding. These episodes introduce:

- how to find the average gradient between two points on a function (strongly emphasized as a technique to be practised);
- the idea that a quadratic function can be used to model "real" data;
- how to find gradient at a point using differential calculus when a function is known;
- the validity of calculations and indeed the choice of model should be interpreted and considered in the light of the context under consideration.

Here, in this particular lesson, the teacher's 'social' narrative focuses almost exclusively on his imagined growth of the worms in his garden. He uses this, at times with a touch of humour, as general motivation but at other times to motivate/introduce the mathematical processes of interpretation and validation of the model introduced for the growth of the worms (consider, for example, how he suggests that his worms should not be shrinking after eight years as the model might suggest). We suggest that the stories the teacher spins about his worms require more attention than just focusing on the 'social': the mathematical is at times intricately interwoven with this..

In our description of the following lesson we change the narrative device from that adopted in describing and discussing the first lesson in which we separated description and comment. Having established the features of our framework of analysis we now merge description and comment as we describe a very different narrative form used by a teacher in another of our case study colleges.

Lesson 2: representing data fairly

This lesson, in a different college in the north of England, is near the start of the AS Statistics module. In contrast to the previous lesson the classroom is spacious with clusters of tables ensuring that students sit in groups to encourage the 'sociable' classroom practices that are central to the teacher's pedagogic beliefs. In response to being questioned about the practices she employs in her classroom, the teacher is keen to point out that she would like the experience of students to be somewhat different to those likely to be met by students in typical AS Mathematics lessons:

> "I want to get students to think about the maths; I want students to understand; I want students to connect ideas together, to see all those things that go together and I don't

think a text book did that … . So my students don't do a lot of formal written work in the lesson, so that's probably a difference. I mean obviously they do need to practice stuff but they'll go home and practice something… and you know sometimes we will do a little bit of written work, we don't not do it but… its not me standing at the front, how do you do it and right now you practice…I don't like that."

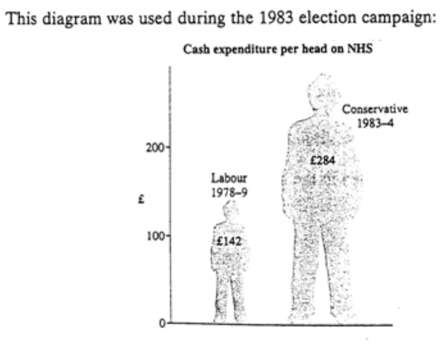
The walls are covered with posters that the teacher's students, including from this class, have made about the mathematics that they have been doing and learning. These are working documents, rather than being carefully polished artefacts primarily for display: students tell us that they refer to these posters to assist them with their learning during lessons. There are interactive and traditional whiteboards at the front of the room.

In the first phase of the lesson the teacher immediately engaged the whole class introducing a 'social' narrative about a forthcoming world cup football match involving England:

> "I've put a couple of blocks up there really. What they represent are, well, there's a certain football match coming up soon, isn't there? Yesterday I actually had it as England against Germany but today it's going to have to be England against Ecuador, isn't it? Ok. So all this does, it's just a visual representation of the number of goals that England's going to score. Ok?"

Having engaged students briefly with a clearly motivating social discourse about a hot topic of the moment the teacher orchestrated a whole class discussion about how they thought they should interpret the diagram. An initial suggestion of 2-1 was quickly followed by a further suggestion of 4-1 with a student recognising that four of the smaller rectangles would fit into the larger rectangle. After a short time the teacher summarised the argument as being, "a debate of whether we are going to go for scale factor [of length] or area."

In the next phase of the lesson the 'social' narrative developed whereas the underlying mathematical narrative remained the same. The teacher introduced a political dimension in the 'social' strand by asking if one of two diagrams she introduced, that had been used to represent change in spending on health, had been used fairly by a political party during an election campaign; she also asked the class to consider if the diagram showing the sale of buns was valid (see Figure 2, below). However, the nature of the ensuing discussion, initially in pairs and then involving the whole class together, was such that the teacher's 'social' and mathematical strands of narrative were closely intertwined in contrast to the narrative of Lesson 1, in which for periods of time the teacher seemed to focus either on one strand or the other.



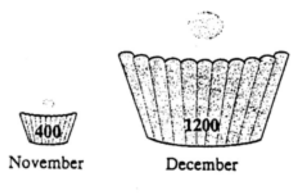


Figure 2: Diagrams that exaggerate frequency

There followed a phase of the lesson in which there was a distinct change in mathematical narrative. Continuing with the pedagogic practice of whole class discussion this focussed on whether a selected range of statistical measures and diagrams were suitable to represent a list of discrete “data”: Up to this point of the lesson the data used had been connected with reality, but this was “data” without connection to context; in reality this was merely a list of integer values. Judicious choice of data and measures allowed the teacher to make another key point: that to be able to use some representations data firstly needs to be sorted into groups. To ensure that students would focus on this, although the teacher suggested to the group that she was drawing on all of the statistical measures and diagrams that they had identified in their posters in the previous lesson, she had in fact carefully selected the list of possible measures/diagrams: averages, quartiles, bar chart, pie chart, box & whisker diagram and scatter graph. Students were asked to consider if there was anything that was problematic from the list. One student suggested the scatter graph and another suggested pie charts, “because the numbers are too diverse”. Another suggested that the data would need putting into groups to allow the drawing of a reasonable bar or pie chart which emphasised the key mathematical idea that the teacher wished to introduce in this phase of the lesson,

> *“That’s what we are going to look at this morning where we put some data into groups ….[and]…. make sure we don’t get misleading diagram at the end of it.”*

The next phase of the lesson was devoted to collecting some data that would be meaningful to the students: finding the error in their estimate of the length of a wiggly line that the teacher had drawn on the board. After the length of the line had been found all students calculated the difference between this and their estimated length with the teacher deliberately contributing a large value ensuring that the data set included an outlier. Overall, we classify the narrative at this stage as being ‘social’, leading as it did to social interaction, with its emphasis being on setting the scene for the further development of both ‘social’ and mathematical narrative strands.

Before the students worked any further with this data the teacher by developed the mathematical narrative by modelling how the students might eventually process and develop a diagrammatic representation of the data by displaying similar data (although different in magnitude) that had been collected by another group and her representation of this as a bar chart/histogram. Again, in whole-class discussion, by judicious questioning, the teacher directed attention to key features of histograms, although at this stage she had not introduced this terminology. For example, the group were firstly asked to consider why there were no gaps between the bars. One student suggested that this was because it was a histogram, clearly having already met the key mathematical object that was to emerge from the lesson. Another suggested it was because the data was in groups. At this stage the social narrative that had introduced the data to be considered was closely aligned with and allowed the mathematical narrative, that now focused on features of representing continuous data by a bar chart, to develop. In continued questioning one of the more able students in the group offered an explanation that it was because the data “share boundaries”.

The lesson moved to the next phase in which there was further focussing on these key ideas as students, again working in pairs, were asked to consider which of four different diagrams most fairly represented the data that the teacher had introduced (see Figure 3). Again in this phase the students were able to, with relative ease, develop their mathematical understanding by making connections with the social narrative which had been carefully designed to facilitate this.

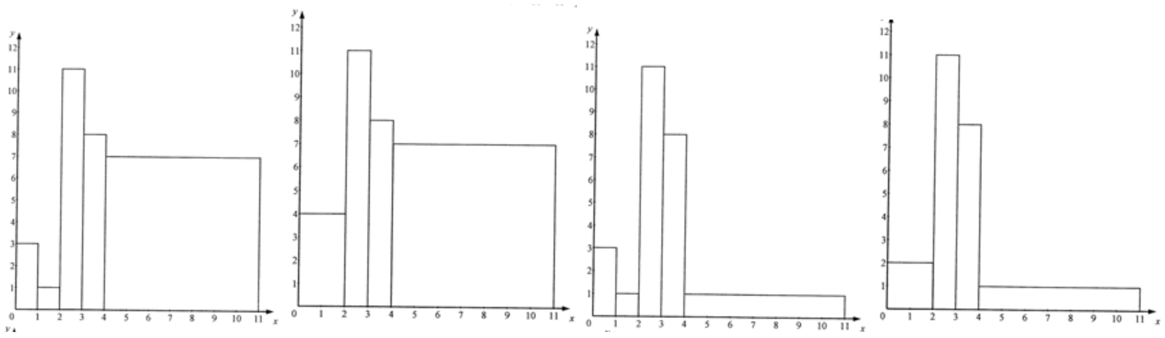


Figure 3: Possible histograms

The teacher circulated from table to table discussing with students their thinking which she was able to draw upon in the whole-class discussion which followed. This phase had a strong mathematical narrative which focussed on the idea that in a "histogram" the area of the bars should be proportional to the frequency that they represent. Firstly the teacher drew attention to the fact that groups had rejected the diagrams with large blocks of height 7 in the interval 4 - 11, and established that although there was a frequency of 7 in this interval it is potentially misleading to have a block of this height across the entire interval. Likewise for the diagram with a single block of height 4 for the interval 0 – 2. When asked to explain why the bar in this interval has a height of 2, a student suggested, "the total height is the mean of the total height"…..going on to explain, "three plus one is four and the mean of that is two…" The teacher asked why this was divided by two and the student explained, "because that's how many columns you have in the width." The teacher assisted this student towards developing a procedure to find the height of a bar, "so you're dividing by the width of the block, aren't you?" Having established this the teacher turned the question round asking where the 7 people were in the preferred histogram and following the suggestion of "area" by a student the teacher summarised the key features of histograms that had emerged,

> *"it's like a bar chart but there's no gaps, but we've also got to have different widths when we've got area representing...so if you're going to put groups together ...you work out how many are in those groups but you've then got to divide by the width to get the correct height."*

In the penultimate phase of the lesson the teacher set the task on which the students would work in pairs in the final phase: she set the challenge of producing a poster of a histogram of their wiggly line data leaving each pair to decide how the data should be grouped. Here, as the teacher re-capped the main ideas of the lesson, the mathematical narrative reached its conclusion as she and then the students brought together the earlier episodes of mathematical narrative, namely that:

- in diagrammatic representations of data area should be proportional to frequency
- some diagrammatic representations of data require the data to be grouped
- "bar charts" (histograms) for continuous data have a continuous horizontal axis and no gaps between the bars.
- the height of the bar of a histogram can be found by dividing the frequency by the width of a bar.

Before pairs of students set off to work on the task the teacher directed attention to how they should work out the correct height of the bars asking how high a bar would be for the 4 people between 0 and 50. Although she referred back to the earlier example having 7 people in a group of width 7 and 4 people in a group of width 2, the students struggled to explain so the teacher illustrated the misleading version (width 4 and height 50) suggesting that this would represent 200 people. There was little response from the whole class at this point and the teacher abstracted a rule from the situation, "number of people divided by the width" the method of "averaging out" suggested by one of the group. In this phase the mathematical narrative was strongly situated in and aligned with the prior 'social' narrative in which bars of the histogram had maybe become fused with the people they represent, much in the way that Roth and Bowen (2001) and Williams and Wake (2007) suggest that workers fuse their mathematics with what it is used to represent in their workplace. Indeed, at this stage of the lesson, we were reminded of the construct of situated abstraction, that Pozzi, Noss and Hoyles (1998) introduce to explain how workers can be thought to partially abstract mathematics from situations that are meaningful to them. Here, the students work with the histograms in terms of people; a next step will be to generalise this in purely mathematical terms using terminology and abstract thinking related to frequency, class width, frequency density and so on.

Conclusion

Lessons viewed as pedagogic events can be analysed as an interweaving of two strands into **a single narrative** construction:

i. a mathematical strand, which is often in effect the development of a mathematical argument, and
ii. a social strand which comprises of social activities (which arise from the teacher's choice of pedagogic practices) and social discourse.

The mathematical strand is driven by the mathematical argument that the teacher wants to present and reflects the way in which the teacher understands how mathematical ideas and processes familiar to his or her students may be (re-) introduced and interconnected to develop new (to the pupils) mathematics. On the other hand, the social strand contains references to 'why' as the teacher draws on a range of practices and discourse with which he or she attempts to motivate and engage his or her students in learning.

Here, in two lessons, we exemplify how the operation of two different teachers in their classrooms can be analysed in this way. Both social and mathematical strands demonstrate the key features of narrative as the teacher engages the class in the development of a unique revelation of new mathematics. The 'social' narrative, in particular, we suggest is extremely important in engaging the learners not only through different activities but also a discourse that can motivate learning and which can also ensure, to a greater or lesser extent, connectivity with the mathematics itself. This appears potentially important as students attempt at the time, and later during periods of reflection, perhaps as they practise newly learnt techniques and so on, to make sense of the place of this new mathematics in the grand scheme of the discipline. Indeed, in Lesson 2 we observe that, much as in performance art, the audience (in this case the students), become part of the narrative themselves: the social strand of narrative ensures they are fully incorporated into the development of the mathematics itself. Importantly this is planned by the teacher from the outset, who although having key episodes in the mathematical development that she wishes to ensure are 'revealed' as part of the overall narrative, is willing to be flexible in this regard to ensure that the students themselves co-construct the narrative of the lesson. This contrasts, on the other hand, with Lesson 1, where the interaction of the students with the narrative is weak: although the teacher introduces a relatively prominent social strand, relating to his worms, he does not incorporate pedagogic practices that might ensure his students are engaged with this. Consequently, we argue, they may not be actively engaged with the mathematical strand to the narrative.

Epilogue: a word of caution

So you have now completed your reading of our story, or have you? We pose this as a question as perhaps you have, as many do when reading a novel, skipped to the end before ploughing through the narrative as it was conceived by the author(s). Or perhaps you skimmed or skipped some of the episodes. Although narrators carefully craft the stories they tell, particularly when they have the opportunity to reflect on, and refine early versions, their audience does not necessarily engage with the resulting narrative in the way that was intended. This is true for the teacher as narrator: although it is not possible for students to move backward and forward between episodes at will it is possible for them to choose to engage with episodes in ways different to that intended. For example, a student may choose to disengage for parts of a lesson: he may start to day-dream in a particularly long monologue by the teacher, or find it difficult to participate in a group activity due to the social composition of the group. Thus, the "received" narrative has to be 'read', and will have many different interpretations, no matter how much effort the narrator makes to craft their tale. This sets another challenge: that of exploring how learners connect with, and interpret, the different forms of pedagogical events that teachers develop comprising of their mathematical and social strands of narrative.

References

Bernstein B. (1996) *Pedagogy, Symbolic Control and Identity Theory, Research, Critique.* London, Taylor and Francis

Boaler, J. and Greeno, J. (2000). Identity, Agency and Knowing in Mathematics Worlds. J. Boaler (ed) *Multiple Perspectives on Mathematics Teaching and Learning*, (pp. 171-200). Westport, Ablex Publishing.

Bruner, J. (1996). *The Culture of Education*. Cambridge MA., Harvard University Press.

Cooper, B. (1998). Using Bernstein and Bourdieu to understand children's difficulties with "realistic" mathematics testing: an exploratory study. *Qualitative Studies in Education*, 11, 4, pp. 511-32.

Cooper, B. and Dunne, M. (1998). *Social class, gender, equity and National Curriculum tests in maths*. Mathematics Education and Society 1, Nottingham.

Cooper, B. and Dunne, M. (2000). *Assessing Children's Mathematical Knowledge: social class, sex and problem solving*. Buckingham: Open University Press.

Davis, P., Williams, J; Black, L.; Hernandez-Martinez, P.; Nicholson, S.; Pampaka, M. and Wake, G. (2007) Students' mathematical identity and its relation to classroom mathematics social practice. Paper presented at BERA 2007.

DfES (2005). *Improving Learning in Mathematics.* Standards Unit, Teaching and Learning Division.

Freudenthal, H. (1983). *Didactical fenomenology of mathematical structures*. Dordrecht: Reidel

Mor, Y. and Noss, R. (in press) submitted to Special issue on Narrative and Interactive Learning Environment of the *International Journal of Continuing Engineering Education and Life-Long Learning*

Moshovitz-Hadar, N. (1988) School Mathematics Theorems – an endless source of surprise, *For the Learning of Mathematics,* 8, 3, pp.34-40.

Ofsted (2006). Evaluating Mathematics Provision for 14-19-year-olds. London

Pietig, J. (1997) Foundations and teacher education: do we need a new metaphor? *Journal of Teacher Education*, 48 (3), 177-185.

Pozzi, S., Noss, R., & Hoyles, C. (1998). Tools in Practice, Mathematics in Use. *Educational Studies in Mathematics, 36*, 105-122.

Roth, W.-M. and Bowen, G.M.: 2001, Professionals read graphs: A semiotic analysis, *Journal for Research in Mathematics Education,* 32**,** 150-194.

Ryan, J., and Williams, J. Children's Mathematics 4-15. Learning From Errors and Misconceptions. Maidenhead, Open University Press.

Shulman, L. S. (1986)Those Who Understand: Knowledge Growth in Teaching. *Educational Researcher*, 15 (2), pp 4 – 14.

Stigler J. W., and Hiebert, J. (1999) *The Teaching Gap* (2nd ed.) New York: The Free Press

Swan, M. (2006) *Collaborative Learning in Mathematics – A Challenge To Our Beliefs and Practices.* London, NRDC.

Tall, D and West, B. (1986) Graphic insight into calculus and differential equations, in *The Influence of Computers and Informatics on Mathematics and its Teaching* (ed. Howson G. & Kahane J-P), Cambridge, C.U.P. pp. 107-119.

Treffers, A. (1987) *Three Dimensions: A Model of Goal and Theory Description in Mathematics Instruction – The Wiskobas Project.* Dordrecht, Reidel.

Van den Heuvel-Panhuizen, M. (2001) Realistic Mathematics Education as Work in Progress, in *Common Sense in Mathematics Education. Proceedings of The Netherlands and Taiwan Conference on Mathematics Education,* (ed. Lin F.L.) Taipei, Taiwan.

Williams, J. S.., and Wake, G.D., (2007). BlackBoxes in Workplace Mathematics. *Educational Studies in Mathematics*, 64 (3), 317-343.

Wierzbicka, A. (1999). German "cultural scripts": Public signs as a key to social attitudes and cultural values. *Discourse and Society, 9*(2), 241-282.